PHYSICS 110A, WINTER 2017
ELECTRICITY AND MAGNETISM

Assignment One, Due Friday, January 12, 5:00 pm.

Note: There are more simple problems with vectors than the exercises below. Depending on how well you remember this material, you might want to review some of the more basic processes of vector addition, multiplication, etc in addition to this homework.

[1.] Find the angles between (a) the space diagonals of a cube; (b) a space diagonal and an edge; (c) a space diagonal and a diagonal of a face.

[2.] Let $\mathbf{A} = 2\mathbf{x} - \mathbf{y} - \mathbf{z}$, $\mathbf{B} = 2\mathbf{x} - 3\mathbf{y} + \mathbf{z}$, and $\mathbf{C} = \mathbf{y} + \mathbf{z}$. Evaluate $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and $\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$, and show they are equal.

[3.] Let $\mathbf{A} = 2\mathbf{x} - \mathbf{y} + 2\mathbf{z}$. (a) Find the unit vector (length 1) in the same direction as $\mathbf{A}$. (b) Find a vector perpendicular to $\mathbf{A}$. (c) Find a unit vector perpendicular to $\mathbf{A}$.

[4.] Let $\mathbf{A} = 2\mathbf{x} - 3\mathbf{y} + \mathbf{z}$. (a) If $\mathbf{A} \cdot \mathbf{B} = 0$, does it follow that $\mathbf{B} = 0$? If not, find a specific example for $\mathbf{B}$. (b) Answer the same question if $\mathbf{A} \times \mathbf{B} = 0$. (c) Answer the same question if $\mathbf{A} \cdot \mathbf{B} = 0$ *and* $\mathbf{A} \times \mathbf{B} = 0$.

[5.] What is the value of $(\mathbf{A} \times \mathbf{B})^2 + (\mathbf{A} \cdot \mathbf{B})^2$? (Remember the square of a vector is the dot product of the vector with itself.)

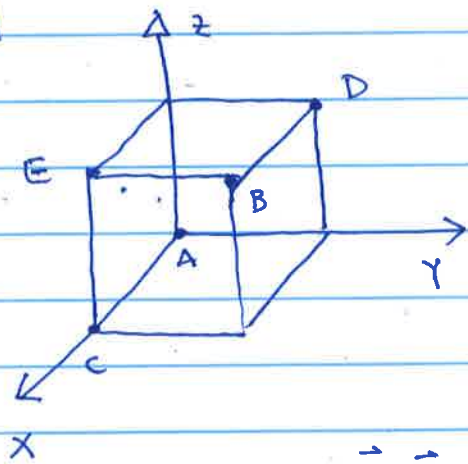
[6.] Prove $(\mathbf{A} \times \mathbf{B}) \cdot \big((\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A})\big)$ is the square of $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

[7.] *Extra Credit (tricky!):* Use the notation and identities in pages V5A-V5B of the notes to prove the result of problem **[6.]**.

1.

P110A PS1 Solutions

$\boxed{1}$



a) SPACE DIAGONALS:

A: (0,0,0) to B: (1,1,1)

$$\vec{v} = (1,1,1)$$

C: (100) to D: (0,1,1)

$$\vec{w} = (-1,1,1)$$

$$\vec{v}\cdot\vec{w} = v_x w_x + v_y w_y + v_z w_z = 1 = |\vec{v}||\vec{w}|\cos\theta_{vw}$$

$$|\vec{v}| = \sqrt{3} \qquad |\vec{w}| = \sqrt{3}$$

$$\rightarrow \cos\theta_{vw} = 1/3 \rightarrow \theta_{vw} = 70.53^\circ$$

b) edge eg A: (000) to C(100) $\vec{u} = (1,0,0)$

$$\vec{v}\cdot\vec{u} = v_x u_x + v_y u_y + v_z u_z = 1 = |\vec{v}||\vec{u}|\cos\theta_{vu}$$

$$|\vec{v}| = \sqrt{3} \qquad |\vec{u}| = 1$$

$$\rightarrow \cos\theta_{vu} = 1/\sqrt{3} \rightarrow \theta_{vw} = 54.73^\circ$$

c) face eg A: (000) to E(1,0,1) $\vec{t} = (1,0,1)$

$$\vec{v}\cdot\vec{t} = v_x t_x + v_y t_y + v_z t_z = 2 = |\vec{v}||\vec{t}|\cos\theta_{vt}$$

$$|\vec{v}| = \sqrt{3} \qquad |\vec{t}| = \sqrt{2}$$

$$\rightarrow \cos\theta_{vt} = \sqrt{\frac{2}{3}} \rightarrow \theta_{vt} = 35.26^\circ$$

2.

$\boxed{2}$ $\vec{A} = 2\hat{x} - \hat{y} - \hat{z}$

$\vec{B} = 2\hat{x} - 3\hat{y} + \hat{z}$

$\vec{C} = \hat{y} + \hat{z}$

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & -3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -4\hat{x} - 2\hat{y} + 2\hat{z}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & -1 & -1 \\ -4 & -2 & 2 \end{vmatrix} = \hat{x}(-4) + \hat{y}(0) + \hat{z}(-8)$$

$\vec{A} \cdot \vec{C} = -2$ $\vec{A} \cdot \vec{B} = 6$

$\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$= (2\hat{x} - 3\hat{y} + \hat{z})(-2) - (\hat{y} + \hat{z})(6)$

$= -4\hat{x} + 0\hat{y} - 8\hat{z}$ ✓✓

3.

$\boxed{3}$ a) $\vec{A} = 2\hat{x} - \hat{y} - \hat{z}$

$|\vec{A}| = \sqrt{6}$

unit vector $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|} = \dfrac{1}{\sqrt{6}}(2\hat{x} - \hat{y} - \hat{z})$

b) $\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$

$\vec{B} \perp \vec{A} \Rightarrow \underbrace{\vec{A}\cdot\vec{B} = 0}$

$2B_x - B_y - B_z = 0$

many choices eg $B_x = 1 \quad B_y = 2 \quad B_z = 0$

ie $\vec{B} = \hat{x} + 2\hat{y}$

unit vector $\hat{B} = \dfrac{1}{\sqrt{5}}(\hat{x} + 2\hat{y})$

$\boxed{4}$ a) $\vec{A} = 2\hat{x} - 3\hat{y} + \hat{z}$

$\vec{A} \cdot \vec{B} = 0$ only means $\vec{B} \perp \vec{A}$ not that $\vec{B} = 0$

Same argument as #3 $\Rightarrow$ $2B_x - 3B_y + B_z = 0$

so for example $B_y = 1$ $B_z = 3$ $\vec{B} = \hat{y} + 3\hat{z}$

b) $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & -3 & 1 \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}(-3B_z - B_y)$

$+ \hat{y}(B_x - 2B_z)$

$+ \hat{z}(2B_y - 3B_x)$

if $\vec{A} \times \vec{B} = 0$ we need
$$\left.\begin{aligned} -3B_z - B_y &= 0 \\ B_x - 2B_z &= 0 \\ 2B_y + 3B_x &= 0 \end{aligned}\right\} \text{3 homogeneous eqns in 3 unknowns}$$

Soln is $B_x = B_y = B_z$ unless eqns are linearly dependent!

They are linearly dependent $\leftarrow$

since $2 \times$(top eqn) + bottom eqn

is $-6B_z + 3B_x = 0$

$-2B_z + B_x = 0$ $\leftarrow$ same as middle eqn

But we know this had to happen because we know all $\vec{A} \times \vec{B} = 0$ means is that $\vec{B} \parallel \vec{A}$ not that $\vec{B} = 0$

$$\left.\begin{aligned} &\text{top eqn} & B_y &= -3B_z \\ &\text{middle eqn} & B_x &= 2B_z \end{aligned}\right\} \rightarrow \vec{B} = B_z(2, -3, 1)$$

$\nearrow \vec{A}$!

5.

$\boxed{4}$ cont'd

So we have explicitly proven that $\vec{B} \parallel \vec{A}$!

c) If we demand $\vec{B} \parallel \vec{A}$ and $\vec{B} \perp \vec{A}$ we must have $\vec{B} = 0$

This is obvious but can also see via

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta_{AB} \qquad |\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta_{AB}$$

$\sqrt{14}$ $\qquad$ $\sqrt{14}$

The only way $|\vec{B}|\cos\theta_{AB} = 0$ and $|\vec{B}|\sin\theta_{AB} = 0$

is if $|\vec{B}| = 0$.

$\boxed{5}$ $$(\vec{A} \times \vec{B})^2 = |\vec{A} \times \vec{B}||\vec{A} \times \vec{B}| \cos\theta_{A\times B, A\times B} = |\vec{A} \times \vec{B}|^2 = |\vec{A}|^2 |\vec{B}|^2 \sin^2\theta_{AB}$$

$$(A \cdot B)^2 = |\vec{A}|^2 |\vec{B}|^2 \cos^2\theta_{AB}$$

$$\therefore \quad (\vec{A} \times \vec{B})^2 + (\vec{A} \cdot \vec{B})^2 = |\vec{A}|^2 |\vec{B}|^2$$

$\boxed{6}$ $(\vec{A}\times\vec{B})\cdot\left(\underbrace{(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A})}_{\text{this is } \vec{D}\times(\vec{C}\times\vec{A}) \text{ with } \vec{D}=\vec{B}\times\vec{C}}\right)$

$$\underbrace{\vec{C}\,(\vec{D}\cdot\vec{A})} - \vec{A}\,(\vec{D}\cdot\vec{C})$$

but $\vec{D}\cdot\vec{C}=0$
since $\vec{D}=\vec{B}\times\vec{C}$ and
hence $\vec{D}\perp\vec{C}$

$\therefore$ $\underbrace{(\vec{A}\times\vec{B})\cdot\vec{C}}_{\text{volume of parallelopipad so also equals } \vec{A}\cdot(\vec{B}\times\vec{C})}\left(\underbrace{(\vec{B}\times\vec{C})\cdot\vec{A}}_{\text{a \#}}\right)$

so $\left[\vec{A}\cdot(\vec{B}\times\vec{C})\right]^2$ ∎

7

From the notes:

$\boxed{7}$ Dot product of 2 vectors $\vec{A}\cdot\vec{B} = A_i B_i$

rule#1 ↗ ↗ sum over i understood

rule#2 }

i th component of $\vec{A}\times\vec{B}$ is $\epsilon_{ijk} A_j B_k$

Applying these facts

$$(\vec{A}\times\vec{B})\cdot\left[(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A})\right]$$

rule#1

$$(A\times B)_i \left[(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A})\right]_i$$

rule#2

$$\epsilon_{ijk} A_j B_k \; \epsilon_{i\alpha\beta} (\vec{B}\times\vec{C})_\alpha (\vec{C}\times\vec{A})_\beta$$

$$\epsilon_{ijk}\epsilon_{i\alpha\beta} = \delta_{j\alpha}\delta_{k\beta} - \delta_{j\beta}\delta_{k\alpha} \leftarrow \text{rule\#3}$$

Applying rule#3

$$A_j B_k \left[(\vec{B}\times\vec{C})_j (\vec{C}\times\vec{A})_k - (\vec{B}\times\vec{C})_k (\vec{C}\times\vec{A})_j\right]$$

In this case the ϵ_{ijk} made life harder than #6 proof. However, it is often <u>easier</u> to do things with ϵ_{ijk}.

$$B_k(\vec{B}\times\vec{C})_k = \vec{B}\cdot(\vec{B}\times\vec{C}) = 0$$

so second term vanishes

$$A_j(\vec{B}\times\vec{C})_j = \vec{A}\cdot(\vec{B}\times\vec{C}) \quad \text{(rule\#1)}$$

$$B_k(\vec{C}\times\vec{A})_k = \vec{B}\cdot(\vec{C}\times\vec{A}) \quad \text{(rule\#2)}$$

$$\| $$

$$\vec{A}\cdot(\vec{B}\times\vec{C})$$

so again we prove the result.