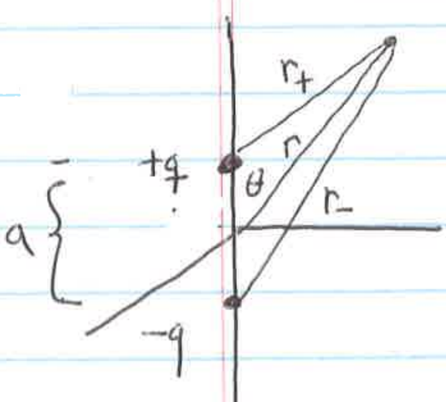
MP1

The sol'n to Laplace's Eqn in spherical coordinates as expansion in Legendre polynomials allows for a classification of $\vec{E}$ fields based on how rapidly they fall off with $1/r$

$\boxed{\text{point charge}}$ $$V(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0} q/r$$

$\boxed{\text{dipole}}$ $$V(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0}\left\{ 1/r_+ - 1/r_- \right\}$$



$$= \frac{q}{4\pi\epsilon_0}\left\{ \frac{1}{\sqrt{r^2 + a^2/4 - ar\cos\theta}} - \frac{1}{\sqrt{r^2 + \frac{a^2}{4} + ar\cos\theta}} \right\}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{r}\left\{ \left(1 - a/r\cos\theta + a^2/4r^2\right)^{-1/2} - \left(1 + a/r\cos\theta + a^2/4r^2\right)^{-1/2} \right\}$$

$$V(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0 r}\left\{ 1 + \frac{a}{2r}\cos\theta - \frac{a^2}{8r^2} + \frac{3}{8}\frac{a^2}{r^2}\cos^2\theta - 1 + \frac{a}{2r}\cos\theta + \frac{a^2}{8r^2} - \frac{3}{8}\frac{a^2}{r^2}\cos^2\theta \right\}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{a\cos\theta}{r^2} + \ldots$$

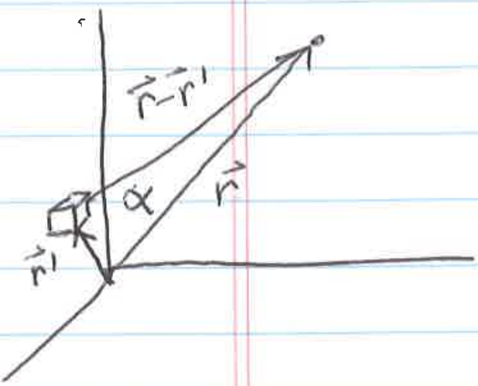
$$\frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

$|\vec{p}| = aq$

MP2

Generally

$$V(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')\,d\tau'}{r}$$



$$= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')\,d\tau'}{[r^2 - 2rr'\cos\alpha + r'^2]^{1/2}}$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')\,d\tau'}{r\,[1 - 2r'/r\,\cos\alpha + r'^2/r^2]^{1/2}}$$

$$V(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0 r}\underbrace{\int \rho(r')\,d\tau'}_{q_{tot}} + \frac{1}{4\pi\epsilon_0 r^2}\underbrace{\int \rho(r')\,r'\,d\tau'}_{\text{dipole moment}}\cos\alpha + \ldots$$

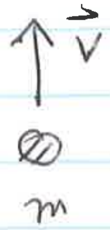
$$V(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\,\rho(r')\,d\tau'$$

"Multipole expansion of potential"

If choose $\hat{z}$ axis along direction $\vec{r}$ to field point then $\alpha \to \theta$

HW: Think about how choice of origin affects the various terms.

Choice of origin in other contexts



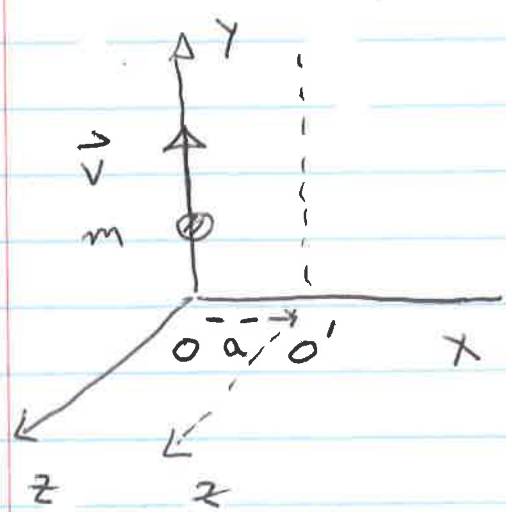
What is <u>linear</u> momentum?

$$\vec{p} = m\vec{v}$$

do not need to specify origin.

What is angular momentum?

$$\vec{r} \times \vec{p} = y\hat{y} \times mv\hat{y}$$



$\vec{L} = \emptyset$ with original origin

but $\vec{L} = (-a\hat{x} + y\hat{y}) \times mv\hat{y}$

$= -mva\hat{z}$

"clockwise"

on the other hand, torque

$$\vec{\tau}' = (\vec{r} + \vec{c}) \times \vec{F} = \vec{\tau}' + \vec{c} \times \vec{F}$$

$$\vec{L}' = (\vec{r} + \vec{c}) \times \vec{p} = \vec{L} + \vec{c} \times \vec{p}$$

$$\frac{d\vec{L}'}{dt} = \frac{d\vec{L}}{dt} + \vec{c} \times \frac{d\vec{p}}{dt} = \underbrace{\frac{d\vec{L}}{dt}}_{\vec{\tau}} + \vec{c} \times \vec{F} = \vec{\tau}'$$

so $d\vec{L}/dt = \vec{\tau}$

$d\vec{L}'/dt = \vec{\tau}'$

MP4

$$V_{mon} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \qquad V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\sum \hat{r}_i p_i}{r^2} \leftarrow \hat{r}\cdot\vec{p}$$

$$\vec{p} = \int \vec{r}\,'\rho(r')d\tau'$$

Quadrupolar term

$\hat{r}_i \equiv$ ith component of $\hat{r}$

$$V_{quad} = \frac{1}{4\pi\epsilon_0}\frac{\sum \hat{r}_i\hat{r}_j Q_{ij}}{r^3}$$

$$Q_{ij} = \frac{1}{2}\int\left(3\hat{r}_i'\hat{r}_j' - r'^2\delta_{ij}\right)\rho(r')d\tau'$$

proof

$$\sum_{ij} \hat{r}_i\hat{r}_j Q_{ij} = \frac{1}{2}\int\left\{3\underbrace{\sum_i \hat{r}_i r_i'}_{\hat{r}\cdot\vec{r}\,' = r'\cos\alpha}\underbrace{\sum_j \hat{r}_j r_j'}_{r'\cos\alpha} - r'^2\underbrace{\sum_{ij}\hat{r}_i\hat{r}_j\delta_{ij}}_{\hat{r}\cdot\hat{r}=1}\right\}\rho(r')d\tau'$$

$$= \int r'^2\underbrace{\frac{1}{2}(3\cos^2\alpha - 1)}_{P_2(\cos\alpha)}\rho(r')d\tau'$$

MP4A

Analogy to classical Mechanics and moment of inertia

$\vec{L} = I\vec{\omega}$

$\vec{L}$ need not be parallel to $\vec{\omega}$!

$$\vec{\tau} = I\vec{\alpha} = I\, d\vec{\omega}/dt$$ is incomplete

$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} d\omega_1/dt \\ d\omega_2/dt \\ d\omega_3/dt \end{pmatrix}$$

This is what makes rotation problems so incredibly complicated and weird.
(precession of spinning top, nutation, etc)

Moment of inertia "tensor"

$$I_{ij} = \frac{1}{2}\int \left(3 r_i' r_j' - (r_i')^2 \delta_{ij}\right] \rho(r')\, d\tau'$$

mass density rather than charge density.