Laplace Eqn in Spherical Coordinates and Legendre polynomials

We know $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$

is potential due to point charge Q at origin

obeys $\nabla^2 V = 0$ Laplace eqn $\vec{r} \neq 0$

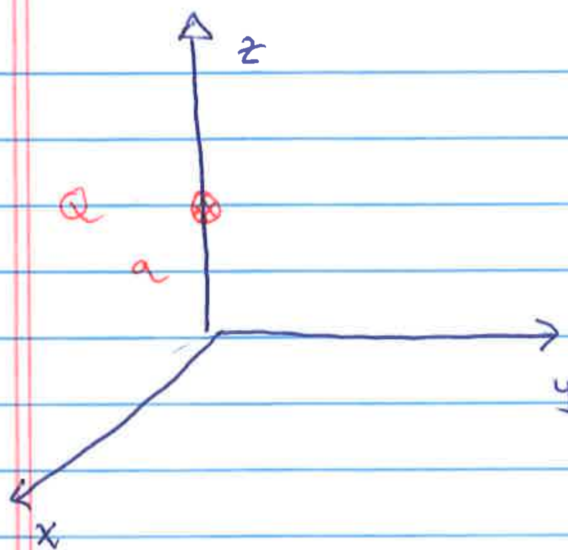
(and $\nabla^2 V = -Q/\epsilon_0\,\delta(\vec{r})$ more generally)

We want to find other solutions for more general charge distributions. (We already know a few others like spherical shell, sphere of constant ρ, etc...)

We also know how to get $V(\vec{r})$ along special directions like z-axis.

Consider point charge Q a bit away from origin, along $\hat{z}$ axis

L2



Obviously

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2+y^2+(z-a)^2}}$$

$$x^2+y^2+(z-a)^2 = x^2+y^2+z^2-2za+a^2$$

$$= r^2 - 2ra\cos\theta + a^2$$

$$V(r,\theta,\phi) = \frac{Q}{4\pi\epsilon_0 r}\left(1-\frac{2a}{r}\cos\theta + a^2/r^2\right)^{-1/2}$$

no ϕ dependence

Binomial (okay even if $n \neq$ integer)

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3 + \dots$$

$$\left(1-\frac{2a}{r}\cos\theta+\frac{a^2}{r^2}\right)^{-1/2} = 1 - \frac{1}{2}\left(-\frac{2a}{r}\cos\theta + \frac{a^2}{r^2}\right)$$

$$+ \frac{3}{8}\left(-\frac{2a}{r}\cos\theta + a^2/r^2\right)^2$$

$$- \frac{5}{16}\left(-\frac{2a}{r}\cos\theta + a^2/r^2\right)^3 + \dots$$

L3

Powers of a/r

$(a/r)^0$	1
$(a/r)^1$	$\cos\theta$
$(a/r)^2$	$-1/2 + 3/2 \cos\theta$
$(a/r)^3$	$-3/2 \cos\theta + 5/2 \cos^2\theta$
$(a/r)^4$	$3/8 - 30/8 \cos^2\theta + 35/8 \cos^4\theta$

$\vdots$

Legendre polynomials $p_n(\cos\theta)$ are coefficients of $(a/r)^n$

$p_0(x) = 1$

$p_1(x) = 3/2\, x - 1/2$

$p_2(x) = 5/2\, x^2 - 3/2$

$p_3(x) = 35/8\, x^4 - 30/8\, x^2 + 3/8$

Notice some things $p_n(1) = 1 \quad \forall n$

$p_n(x) = p_n(x) \quad n$ even

$-p_n(x) \quad n$ odd

At this point...

Harder to see except by calculating

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n+1} \delta_{nm}$$

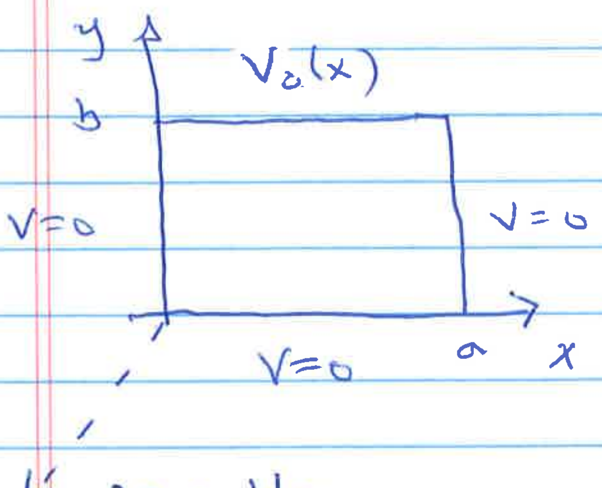
Will see why later, from several points of view

$\boxed{Q}$ Can you guess one Already?

L5

VARIANT OF PREVIOUS

Review Rectangular laplace $V(x,y) = f(x)g(y)$



$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$g\frac{d^2 f}{dx^2} + f\frac{d^2 g}{dy^2} = 0$$

$$-\frac{1}{f}\frac{d^2}{dx^2}f = \frac{1}{g}\frac{d^2 g}{dy^2} = k^2$$

2D problem
or very long in $\hat{z}$
direction

$f(x) = \sin kx, \cos kx$

$k = \frac{n\pi}{a}$
since
$V=0$ @ $x=a$

no since
$V=0$ @ $x=0$

$g(y) = e^{ky}, e^{-ky}$

$\cosh ky, \sinh ky$

no since $V=0$
@ $y=0$

$$V(x,y) = \sum_n A_n \sinh\frac{n\pi y}{a}\sin\frac{n\pi x}{a} \Rightarrow V_0(x) \text{ at } y=b$$

$$\sum_n A_n e^{n\pi b/a}\sin\frac{n\pi x}{a} = V_0(x)$$

"DOT PRODUCT WITH $\sin\frac{m\pi x}{a}$" $\rightarrow$

$$A_m e^{m\pi b/a}\frac{a}{2} = \int_0^a V_0(x)\sin\frac{m\pi x}{a}dx$$

$$\int_0^a \sin\frac{m\pi x}{a}dx$$

$\rightarrow \delta_{nm}$

L6

$$V(x,y) = \sum_n \underbrace{\frac{2}{a} e^{-n\pi b/a} \int_0^a V_0(x') \sin\frac{n\pi x'}{a} dx'}_{A_n} \sinh\frac{n\pi y}{a} \sin\frac{n\pi x}{a}$$

$\boxed{\text{NEW}}$

$$= \int_0^a dx' \, V_0(x') \underbrace{\left\{ \sum_n \frac{2}{a} e^{-n\pi b/a} \sin\frac{n\pi x'}{a} \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} \right\}}_{\equiv G_0(x, x', y)}$$

$G_0(x,x',y)$ propagates $V_0(x')$ to $V(x,y)$

"Green's function or propagator"

G depends on PDE (Laplace) and bdy conditions.
in our case

STRATEGY PDE $\leftrightarrow$ Laplace

BC

$\downarrow$

Separate variables

$\downarrow$

ODE $\rightarrow$ "special functions"

sin, cos

sinh, cosh

L7

Legendre, Spherical harmonics

Same procedure for Laplace in spherical coordinates or cylindrical coordinates except special functions less familiar

Bessel

State result for spherical.

(1) No ϕ dependence

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Legendre polynomials

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$

L7A

Examples

(1) Point charge at origin $\quad V(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$

$$A_\ell = 0 \quad \forall \ell$$

$$B_0 = Q/4\pi\epsilon_0 \quad \ell = 0$$

$$B_\ell = 0 \quad \forall \ell \neq 0$$

(2) Point charge at $z = a$ $\quad V(r,\theta) = \frac{Q}{4\pi\epsilon_0 r}\frac{1}{\sqrt{1 - 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}}}$

$$= \frac{Q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\left(\frac{a}{r}\right)^\ell P_\ell(\cos\theta)$$

gives $\frac{1}{r^{\ell+1}}$

using generating function on next page

$h = \frac{a}{r}$

$x = \cos\theta$

$$\Rightarrow \quad A_\ell = 0 \quad \forall \ell$$

$$B_\ell = \frac{Q}{4\pi\epsilon_0}a^\ell$$

(3) Laplace with no θ dependence either (MT problem)

$$1/r^2 \, d/dr \, r^2 \, dV(r)/dr = 0$$

$$d/dr \, r^2 \, dV/dr = 0$$

$$r^2 \, dV/dr = -B \qquad dV/dr = -B/r^2$$

$$V(r) = A + B/r$$

ie only $\ell = 0$ are nonzero

L8

Properties of Legendre polynomials

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

Like $\int_0^{a} \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a}\, dx = \frac{a}{2}\delta_{nm}$

$P_\ell(1) = 1$ ← check by example

Rodrigues' formula $P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$

Generating function $\frac{1}{\sqrt{1-2xh+h^2}} = \sum_{\ell=0}^{\infty} h^\ell P_\ell(x)$

PDE Legendre functions obey $(1-x^2)\frac{d^2}{dx^2}P_\ell(x) - 2x\frac{d}{dx}P_\ell(x) + \ell(\ell+1)P_\ell(x) = 0$

$-2x = \frac{d}{dx}(1-x^2) \Rightarrow \mathcal{L}$ is Hermitian

$\Rightarrow P_\ell(x)$ orthogonal

$\exists$ complete

i.e. this arises from separation of variables in Laplace in spherical coordinates

Intertwine these facts. eg can show PDE ⟷ generating function

L7B

See where this comes from with separation of variables

$V(r,\theta,\phi)$ $$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial V}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial V}{\partial\theta} + \cancel{\frac{1}{r^2\sin^2\phi}\frac{\partial^2 V}{\partial\phi^2}} = 0$$

$$V(r,\theta) = R(r)\,Q(\theta)$$

$$\frac{1}{R}\frac{d}{dr} r^2 \frac{dR}{\partial r} + \frac{1}{Q\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dQ}{d\theta} = 0$$

$$\frac{1}{R}\frac{d}{dr} r^2 \frac{dR}{dr} = \ell(\ell+1) = -\frac{1}{Q\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dQ}{d\theta}$$

in cartesian $k^2 \equiv C$
to avoid $\sqrt{C}$
similarly here

$$\frac{d}{dr} r^2 \frac{dR}{dr} = \ell(\ell+1)R$$

$$R = r^\ell$$

$$\frac{dR}{dr} = \ell r^{\ell-1}$$

$$r^2\frac{dR}{dr} = \ell r^{\ell+1}$$

$$\frac{d}{dr} r^2 \frac{dR}{dr} = \ell(\ell+1) r^\ell$$

$$= \ell(\ell+1) R \quad \checkmark\checkmark$$

$V_0(\theta)$ is specified on surface of hollow sphere

Find $V(r,\theta)$ inside

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)$$

$B_\ell = 0$ to keep V finite at $r=0$

At surface $r=R$

$\vec{v} = \sum a_\ell \hat{e}_\ell$

$$V(R,\theta) = \sum A_\ell R^\ell P_\ell(\cos\theta)$$

$$\int_0^\pi \underbrace{V(R,\theta)}_{V_0(\theta)\text{ given}} P_n(\cos\theta) = \sum_\ell A_\ell R^\ell \underbrace{\int_0^\pi P_n(\cos\theta) P_\ell(\cos\theta)\, d\theta}_{\frac{2}{2\ell+1}\delta_{n\ell}}$$

$\hat{e}_n \cdot \vec{v} = a_n$

$$= \frac{2}{2n+1} A_n R^n$$

$$A_n = \frac{2n+1}{2R^n}\int_0^\pi V_0(\theta) P_n(\cos\theta)\, d\theta$$

Example: $V_0(\theta) = k\sin^2\frac{\theta}{2} = \frac{k}{2}(1-\cos\theta) = \frac{k}{2}\left(P_0(\cos\theta) - P_1(\cos\theta)\right)$

$$\therefore \quad A_0 = \frac{1}{2}\ \frac{2}{1}\ \frac{k}{2} = \frac{k}{2}$$

$$A_1 = \frac{3}{2R}\ \frac{2}{3}\ \frac{k}{2} = \frac{k}{2R} \qquad A_\ell = 0 \quad \ell \neq 0,1$$

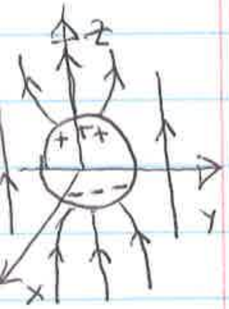
$$V(r,\theta) = \frac{k}{2}\left[1 - \frac{r}{R}\cos\theta\right]$$

L10

$\left\{\begin{array}{l}\bullet \text{ Arises a lot} \\ \bullet\ V \not\to 0 \text{ at } r=\infty\end{array}\right\}$ Useful Example

Conducting sphere in uniform $\vec{E}$ field $\vec{E} = E_0 \hat{z}$

Bdy condition @ $r=\infty$ $\quad V = -E_0 z + C$



Bdy condition @ $r=R$ $\quad V=0$ ← arbitrary choice

This means $V=0$ $\forall z=0$ by symmetry so $C=0$

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

Consider $r\to\infty$ $\quad$ must have $A_2 = A_3 = A_4 = \ldots = 0$

and $A_1 r P_1(\cos\theta) = A_1 r\cos\theta = A_1 z$

Thus $A_1 = -E_0$

at $r=R$ $\quad V(R,\theta) = 0 = \sum_{0}^{\infty}\left(A_\ell R^\ell + B_\ell / R^{\ell+1}\right) P_\ell(\cos\theta)$

$\Downarrow$

$A_\ell R^\ell + B_\ell / R^{\ell+1} = 0$ $\quad$ for each ℓ since $P_\ell(\cos\theta)$ independent

$B_\ell = -A_\ell R^{2\ell+1}$

$B_2 = B_3 = B_4 = \ldots\ 0$ since corresponding A_ℓ vanish

$B_1 = -A_1 R^3 = +E_0 R^3$

$$V(r,\theta) = -E_0\left(r - R^3/r^2\right)\cos\theta$$

L11

The contribution from charge induced on sphere is

$$E_0 R^3 / r^2 \cos\theta$$

$$\sigma(\theta) = -\epsilon_0 E_n = -\epsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=R} = \epsilon_0 E_0 \left(1 + \frac{2R^3}{r^3}\right)\cos\theta \bigg|_{r=R}$$

$\nearrow$ normal

$$\sigma(\theta) = 3\epsilon_0 E_0 \cos\theta$$

$\leftarrow$ >0 for $\theta < \pi/2$

<0 for $\theta > \pi/2$

as expected

[5.] (continued)
~~(c) A point charge q is now added to the configuration in part (b). The charge is placed a distance z_0 from the plane along a line through the center of the triangle and normal to its plane. Find an integral representation for the total potential and evaluate the first nonvanishing contribution for $|\vec{r}| \to \infty$. Identify and interpret the various terms.~~

[6.] A conducting sphere of radius a has a total charge Q. Of this charge, $Q - Q_E$ is distributed uniformly through the volume of the sphere, while the remaining Q_E is uniformly distributed in a ring of zero thickness around the equator of the surface of the sphere. Assume that Q and Q_E are of the same sign.
(a) By direct integration, find the potential $\phi(\vec{r})$ along the polar axis of the sphere for $|\vec{r}| > a$.
(b) By expressing the potential in a power series in spherical coordinates, and comparing with your answer in (a), determine $\phi(\vec{r})$ for all points $\vec{r}$ such that $|\vec{r}| > a$.
(c) Calculate the electric field for large $|\vec{r}|$ to order $a^2/|\vec{r}|^2$. Then consider a particle of charge q (q and Q are of opposite signs) which is held by electrostatic attraction in a bound orbit around the sphere. Show that such an orbit is stable in *inclination* (the angle between the orbit plane and the equatorial plane of the sphere) only if the inclination is zero. Discuss the gravitational analog of this problem. (Recall the earth has an equatorial bulge.)

6. Nonconducting sphere radius a, total charge Q. $Q - Q_E$ uniform in volume Q_E in thin ring about equator. Q, Q_E same sign.

(a) Find $\varphi(\vec{r})$ on polar axis for $|\vec{r}| > a$

The charge $Q - Q_E$ distributed uniformly in sphere must by ignore d... potential i.e. $Q - Q_E$ / placed i.e.

The ring Q_E gives a potential $Q_E/(r^2+a^2)^{1/2}$

Thus $\varphi(\vec{r}) = \dfrac{Q - Q_E}{r} + \dfrac{Q_E}{(r^2+a^2)^{1/2}}$

where $r = |\vec{r}| =$ distance of point $\vec{r}$ from sphere center

(b) Express potential as power series in spherical coordinates and hence obtain $\varphi(\vec{r})$ $\forall\ \vec{r}$ with $|\vec{r}| > a$ even off polar axis!

$$\varphi(\vec{r}) = \frac{Q - Q_E}{r} + \frac{Q_E}{r}\left(1 + a^2/r^2\right)^{-1/2}$$

$$= \frac{Q - Q_E}{r} + \frac{Q_E}{r}\left(1 - \frac{1}{2}\frac{a^2}{r^2} + \frac{3}{8}\frac{a^4}{r^4} - \dots\right)$$

$$= \frac{Q}{r} + Q_E\left(-\frac{1}{2}\frac{a^2}{r^3} + \frac{3}{8}\frac{a^4}{r^5} - \dots\right)$$

Now according to the prescription in Jackson, we append a factor of $P_\ell(\cos\theta)$ to any term in $1/r^{\ell+1}$,

$$\boxed{\varphi(\vec{r}) = \frac{Q}{r} + Q_E\left(-\frac{1}{2}\frac{a^2}{r^3} P_2(\cos\theta) + \frac{3}{8}\frac{a^4}{r^5} P_4(\cos\theta) + \dots\right.}$$

(c) calculate $\bar{E}$ for large $\vec{r}$ to order a^2/r^2. Consider particle of charge q (of opposite sign to Q) held in bound orbit. Show orbit is stable in inclination, the angle between orbit plane and equatorial plane) only for zero inclination. Discuss grav. analog.

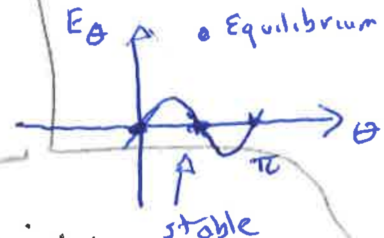
$$\vec{E} = -\nabla\varphi(\vec{r}) = -\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{Q}{r} - \frac{Q_E a^2}{2}\frac{1}{r^3}\frac{1}{2}(3\cos^2\theta - 1) + \dots\right)$$

$$\vec{E} = -\hat{r}\left(-\frac{Q}{r^2} + \frac{3Q_E a^2}{2r^4}(3\cos^2\theta - 1) + \dots\right) + \hat{\theta}\left(+\frac{Q_E a^2}{4r^4}\, 3\, 2\cos\theta\sin\theta + \dots\right)$$

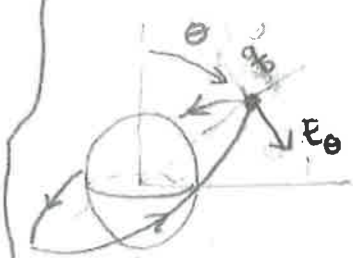
$$\boxed{\vec{E} = \frac{Q}{r^2}\left[1 - \frac{3Q_E}{4Q}\frac{a^2}{r^2}(3\cos^2\theta - 1) + \dots\right]\hat{r} + \frac{Q_E}{r^2}\left[\frac{3\sin 2\theta}{4}\frac{a^2}{r^2} + \dots\right]\hat{\theta}}$$

$\theta = 0$ $|\vec{E}| < Q/r^2$

$\theta = \pi/2$ $|\vec{E}| > Q/r^2$



Now consider the electric field in the $\hat{\theta}$ direction



For the orbit to be stable in inclination, the condition is clearly first that $E_\theta = 0$ so that there be no force moving q out of orbital plane. This condition is satisfied only by $\theta = 0$ and $\theta = \pi/2$.

But for stability we furthermore need that in a small displacement from $\theta = \theta_0$ the force E_θ restores back to θ_0.

use:

$\sin(2d\theta) \sim 2d\theta$

For $\theta = 0$ $E_\theta(d\theta) \simeq Q_E/r^2\left(3\frac{2d\theta}{4}\, a^2/r^2\right)$

This is not stable since E_θ tends to push us further away from $\theta_0 = 0$

$\sin 2(\pi/2 + d\theta)$

$= \sin(\pi + 2d\theta)$

$\sim -2d\theta$

For $\theta = \pi/2$ $E_\theta(d\theta) = Q_E/r^2\left(3\frac{(-2d\theta)}{4}\, a^2/r^2\right)$

This is stable since E_θ tends to push us back towards $\theta_0 = \pi/2$ i.e. E_θ has opposite sign

PHYSICS 200B, WINTER 2017
ELECTRICITY AND MAGNETISM

Assignment Three, Due Friday, February 17, 5:00 pm.

6-3

[1.] A sphere of radius R has a potential $R(r,\theta) = V_0 \cos^2\theta$ on its surface. Determine the potential outside the sphere.

[2.] A sphere of radius R has surface charge density given by $\sigma = \sigma_0 \sin 2\theta \sin\phi$. Determine the potential both inside and outside the sphere. Note: we have not discussed this in class, but it is an easy application of Gauss' Law to relate the discontinuity in the radial derivative of the potential, that is, the radial component of the electric field, to the surface charge density. You will need this.

HW 6-2

[3.] Solve for the potential in the region between two concentric spherical shells of radii a and b, given the potentials $V_a(\theta)$ and $V_b(\theta)$. Your objective should be to write the coefficients of an expansion of the potential as integrals involving the (unknown) functions V_a and V_b. Choose some specific form of the functions that you find particularly amusing (and not too hard!) and do the integrals.

Let's do the simplest problem of this sort, a shell of constant σ. We know the answer. Outside the shell it behaves like a point charge so

$$\underline{\text{outside}} \qquad V(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^2\sigma}{r} = \frac{1}{\epsilon_0}\frac{R^2\sigma}{r}$$

Inside V = constant because $\vec{E}$ vanishes. to match the formula for $r>R$ we have

$$\underline{\text{inside}} \qquad V(r,\theta,\phi) = \frac{1}{\epsilon_0} R\sigma$$

But let's not use this knowledge and instead proceed from the general forms

$$r>R \qquad V(r,\theta,\phi) = \sum_{lm} B_{lm}\, r^{-(l+1)}\, Y_{lm}(\theta,\phi)$$

$$r<R \qquad V(r,\theta,\phi) = \sum_{lm} A_{lm}\, r^{l}\, Y_{lm}(\theta,\phi)$$

At the surface

Gauss' Law:

$$r>R \quad \uparrow E_r^+ = -\frac{\partial V}{\partial r}$$

$$r<R \quad \uparrow E_r^- = -\frac{\partial V}{\partial r}$$

$$\phi_E = (E_r^+ - E_r^-)\cdot A = \underbrace{\sigma}_{} A/\epsilon_0$$

This is a constant and, if decomposed into the independent spherical harmonics, $\therefore$ contains only Y_{00}

$$r>R \qquad E_r^+ = \sum_{lm} (l+1) B_{lm}\, r^{-(l+2)}\, Y_{lm}(\theta,\phi)$$

$$r<R \qquad E_r^- = -\sum_{lm} l\, A_{lm}\, r^{l-1}\, Y_{lm}(\theta,\phi)$$

We conclude, using the orthogonality of the $Y_{\ell m}$

$$\ell \neq 0 \qquad (\ell+1) B_{\ell m} R^{-(\ell+2)} + \ell A_{\ell m} R^{\ell-1} = 0$$

$$\ell = m = 0 \qquad B_{00} R^{-2} = \sigma/\epsilon_0$$

If we also note V must be continuous at $r=R$

$$\sum_{\ell m} A_{\ell m} R^{\ell} Y_{\ell m}(\theta,\phi) = \sum_{\ell m} B_{\ell m} R^{-(\ell+1)} Y_{\ell m}(\theta,\phi)$$

Using the orthogonality of the $Y_{\ell m}$

$$A_{\ell m} R^{\ell} = B_{\ell m} R^{-(\ell+1)}$$

Combining this with the $\ell \neq 0$ eqn the only possible sol'n is

$A_{\ell m} = B_{\ell m} = 0$.

So, we are left with

$$r > R \qquad V(r,\theta,\phi) = \frac{B_{00}}{r} = \sigma R^2/\epsilon_0 r$$

recovering the "obvious" sol'n.

orthogonal!

Spherical Harmonics

$$Y_{00}(\theta,\phi) = \frac{1}{2}\frac{1}{\sqrt{\pi}} \qquad P_0(\cos\theta) = 1$$

$$Y_{1,-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\, e^{-i\phi} \sin\theta$$

$$Y_{10}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta \qquad P_1(\cos\theta) = \cos\theta$$

$$Y_{11}(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\, e^{+i\phi} \sin\theta$$

Y_{2-2}

Y_{2-1}

$$Y_{20} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\cos^2\theta - 1\right) \qquad P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

already has implications for periodic table

$m=0$, $l=0$	$m=0,\pm1$, $l=1$	$m=0,\pm1,\pm2$, $l=2$	
1			1
1	3		4
1	3	5	9

$$\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi \, (Y_{10})^2$$

$$= 2\pi \frac{1}{4}\frac{3}{\pi}\int_0^{\pi} \cos^2\theta \sin\theta\, d\theta$$

$$\frac{3}{2} \quad -\frac{\cos^3\theta}{3}\Big|_0^{\pi}$$

$$2/3$$

$$= 1$$

H He	2
Li Be B C N O F Ne	8
Na Mg Al Si P S Cl Ar	8
K Ca . . . Kr	18

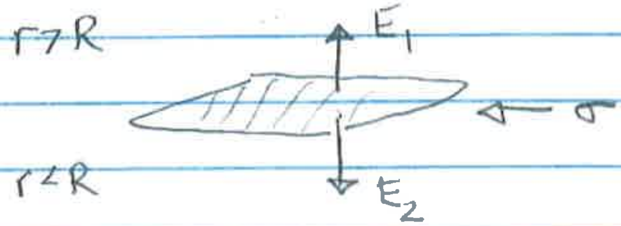
This problem is harder than 1 or 3. The first complication arizes from the lack of azimuthal symmetry so that we are forced to write

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^{l} + B_{lm} r^{-(l+1)}\right] Y_{lm}(\theta,\phi)$$

as the solution to Laplace's Eqn

Inside the sphere we have $B_{lm} = 0$ $\forall\, lm$ to avoid a diverging potential at $r = 0$. Outside the sphere $A_{lm} = 0$ $\forall\, lm$ to be well defined at $r = \infty$.

The next complication is to understand how σ relates the behavior of V for $r<R$ to that at $r>R$. We use Gauss' law



$$\Phi_E = (E_1 A + E_2 A) = \frac{\sigma A}{\epsilon_0}$$

$$E_1 + E_2 = \sigma/\epsilon_0$$

By E_1 and E_2 I really mean the "radial" part of E, ie the part of $\vec{E}$ which contributes to the flux Φ_E.

$$r<R \qquad V(r,\theta,\phi) = \sum_{\ell m} A_{\ell m} r^{\ell} Y_{\ell m}(\theta,\phi)$$

$$r>R \qquad V(r,\theta,\phi) = \sum_{\ell m} B_{\ell m} r^{-(\ell+1)} Y_{\ell m}(\theta,\phi)$$

The simplest $A_{\ell m}, B_{\ell m}$ condition is continuity of V at $r=R$. Thus, together with the orthogonality of the spherical harmonics, implies

$$A_{\ell m} R^{\ell} = B_{\ell m} R^{-(\ell+1)}$$

$$\rightarrow \qquad A_{\ell m} R^{2\ell+1} = B_{\ell m}$$

The more subtle condition is Gauss' law (page 2-1)

$$r<R \qquad E_r = \frac{-\partial V}{\partial r} = -\sum_{\ell m} A_{\ell m}\, \ell\, r^{\ell-1} Y_{\ell m}(\theta,\phi) \qquad (\ell \neq 0)$$

$$r>R \qquad = \sum_{\ell m} B_{\ell m} (\ell+1) r^{-(\ell+2)} Y_{\ell m}(\theta,\phi)$$

whence from page 2-1 we see at $r=R$

$$* \qquad \sum_{\ell m} \left[-A_{\ell m}\, \ell R^{\ell-1} + B_{\ell m}(\ell+1) R^{-(\ell+2)} \right] Y_{\ell m}(\theta,\phi) = \frac{\sigma_0}{\epsilon_0} \sin 2\theta \sin\phi$$

Looking up the forms of $Y_{\ell m}$ we notice

$$Y_{2,1}(\theta,\phi) = -\sqrt{\frac{5}{24\pi}}\, 3\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_{2,-1}(\theta,\phi) = +\sqrt{5/24\pi}\; 3\sin\theta\cos\theta\, e^{-i\phi}$$

But $\quad \sin 2\theta = 2\sin\theta\cos\theta \qquad \sin\phi = 1/2i\,(e^{i\phi} - e^{-i\phi})$

So that $\quad \sin 2\theta \sin\phi = -\sqrt{\frac{24\pi}{5}}\,\frac{2}{3}\,\frac{1}{2i}\left(Y_{2,1} + Y_{2,-1}\right)$

$$= \frac{i}{3}\sqrt{\frac{24\pi}{5}}\left(Y_{2,1} + Y_{2,-1}\right)$$

The orthogonality of the $Y_{\ell m}$ allows us to get * on page 2-2 term by term. If $(\ell, m) \neq (2, \pm 1)$

$$-A_{\ell m}\,\ell R^{\ell-1} + B_{\ell m}(\ell+1) R^{-(\ell+2)} = 0$$

$$A_{\ell m}\,\frac{\ell}{\ell+1}\,R^{2\ell+1} = B_{\ell m}$$

But the continuity of V at $r=R$ gave the same eqn without the $\ell/\ell+1$ factor. The only way these can both be true is if $A_{\ell m} = B_{\ell m} = 0$. Our intuition might have suggested this... if the "source" the surface charge, only involves Y_{21} and $Y_{2,-1}$ the "result", induced potential also only involves these.

2-4

For $(\ell, m) = (2,1)$

$$A_{21} R^5 = B_{21}$$

$$-A_{21}\, 2R + B_{21}\, 3R^{-4} = \frac{\sigma_0}{\epsilon_0} \frac{i}{3} \sqrt{\frac{24\pi}{5}}$$

$$-A_{21}\, 2R + A_{21}\, 3R = \sigma_0/\epsilon_0 \, \frac{i}{3} \sqrt{\frac{24\pi}{5}} = A_{21} R$$

$$= B_{21}/R^4$$

and the same for $(\ell, m) = (2,-1)$.

Thus outside the sphere $r > R$

$\underline{\underline{r > R}}$

$$V(r,\theta,\phi) = \frac{\sigma_0 R^4}{\epsilon_0} \frac{i}{3} \sqrt{\frac{24\pi}{5}} \left\{ Y_{2,1} + Y_{2,-1} \right\}$$

$$= \frac{\sigma_0 R^4}{\epsilon_0} \left\{ -i \sin\theta\cos\theta\, e^{i\phi} + i \sin\theta \cos\theta\, e^{-i\phi} \right\} r^{-3}$$

$$= \frac{\sigma_0 R^2}{\epsilon_0} \underbrace{\frac{R^2}{r^3} \sin 2\theta \sin\phi}_{}$$

$\sigma_0 R^2$ has units of Q

Same angular structure as surface charge density

If you like can compute total charge on sphere

$$Q_{tot} = \int_0^{2\pi} R\, d\phi \int_0^{\pi} R \sin\theta\, d\theta\; 2 \sin\theta\cos\theta \sin\phi = 0$$

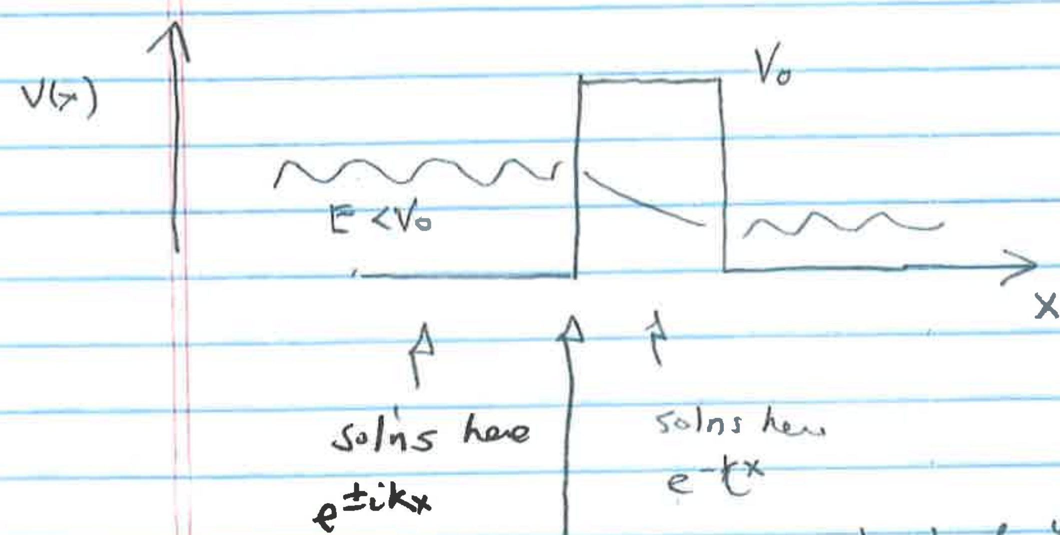
2-5

The fact that $Q_{top} = 0$ is why there is no $1/r$ term in $V(r,\theta,\phi)$ outside sphere.

Obviously the same substitutions can give $V(r,\theta,\phi)$ for $r<R$ and we get

$$\underline{\underline{r<R}} \qquad V(r,\theta,\phi) = \frac{\sigma_0 R^2}{\epsilon_0}\frac{r^2}{R^3}\sin 2\theta \sin\phi \, .$$

Note the similarities of this part of problem with QM problems, eg a particle incident on a potential barrier



must match continuity of $\psi(x)$ and $d\psi/dx$ at bdy. this determines coefficients.