EMG

A natural question: How to divide up field due to "original" charge and the induced polarization ("bound" charges)

original "free" charge

$$\rho = \rho_b + \rho_f$$

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho_b + \rho_f = -\vec{\nabla} \cdot \vec{P} + \rho_f$$

$$\nabla \cdot (\underbrace{\epsilon_0 \vec{E} + \vec{P}}_{\vec{D}}) = \rho_f \qquad \vec{\nabla} \cdot \vec{D} = \rho_f$$

"Electric displacement"

$$\oint \vec{D} \cdot \vec{da} = (Q_f)_{enc} \qquad \text{Gauss' Law}$$

Q: Spherical shell radii a, b with "frozen in" polarization $\vec{P}$ (no free charge) $\vec{P}(r) = k/r\,\hat{r}$. Compute $\vec{E}$

$$\vec{\rho}_b = -\vec{\nabla} \cdot \vec{P} = -1/r^2 \, \partial/\partial r (r^2 k/r) = -k/r^2$$

$$\sigma_b = \vec{P} \cdot \hat{n} = \begin{cases} \vec{P} \cdot \hat{r} = k/b & r = b \\ -\vec{P} \cdot \hat{r} = -k/a & r = a \end{cases}$$

Gauss' Law: $\vec{E}\, 4\pi r^2 = Q_{enc} \hat{r}/\epsilon_0$

$r < a \qquad Q_{enc} = 0 \qquad \vec{E} = 0$

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All from bound charge

$$r > b \qquad Q_{enc} = -k/a \; 4\pi a^2 + k/b \; 4\pi b^2$$

$$+ \int_a^b (4\pi r^2)\,dr\,(-k/r^2)$$

$$= -4\pi k a + 4\pi k b + -4\pi k \underbrace{\int_a^b dr}_{4\pi k(a-b)} = 0$$

Accident!?!

No... our intuition is that total bound charge must be zero because it reflects charge rearrangement of initially neutral object.

General proof

$$Q_{tot} = \oint \vec{\sigma}_b \cdot d\vec{a} + \int (-\vec{\nabla}\cdot\vec{P})\,d\tau = 0 \text{ by divergence Thm}$$

$$a < r < b \qquad Q_{enc} = -\frac{k}{a} 4\pi a^2 + \int_a^r 4\pi r'^2 dr' \left(-\frac{k}{r'^2}\right)$$

$$= -4\pi k a - 4\pi k \underbrace{\int_a^r dr'}_{4\pi k(a-r)} = -4\pi k r$$

Gauss Law

$$4\pi r^2 \vec{E} = -4\pi k r \, \hat{r}/\epsilon_0 \qquad \vec{E} = -k/r\epsilon_0 \, \hat{r}$$

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Another method:

Since $\rho_{free} = 0$ $\quad \vec{D} = 0$ everywhere

$$\therefore \quad \vec{E} = -\vec{P}/\epsilon_0 = \begin{cases} 0 & r<a \\ -\frac{k}{r\epsilon_0}\hat{r} & a<r<b \\ 0 & r>b \end{cases}$$

Much faster!

$\boxed{\text{WARNING}}$ cannot in general compute $\vec{D}$ from a "Coulomb's Law" using ρ_f

$$\vec{\nabla}\cdot\vec{D} = \rho_f \underset{NO}{\Rightarrow} \quad \vec{D}(\vec{r}) \neq \frac{1}{4\pi}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho_f(\vec{r}')\,d\tau'$$

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

in electrostatics $\oint \vec{E}\cdot d\vec{l} = 0 \Rightarrow \vec{\nabla}\times\vec{E} = \phi$

Reason for failure $\quad \vec{\nabla}\times\vec{D} = \epsilon_0 \underbrace{\vec{\nabla}\times\vec{E}}_{\phi} + \vec{\nabla}\times\vec{P} = \vec{\nabla}\times\vec{P}$

But in general $\quad \vec{\nabla}\times\vec{P} \neq 0$

Cannot write $\vec{D}$ as gradient of potential V_f etc.

Example: $\quad \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q\hat{r}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{Q\vec{r}}{r^3}$

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/r^3 & y/r^3 & z/r^3 \end{vmatrix} \qquad \frac{\partial}{\partial y}\frac{z}{r^3} = -3yz/r^5$$

$$= \hat{x}\left(\frac{-3yz}{r^5} - \left(\frac{-3yz}{r^5}\right)\right) + \hat{y} + \hat{z} = \phi$$

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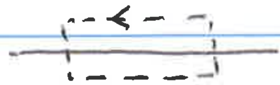
Boundary conditions: ϵ_2 ——— σ_f ϵ_1

$$\oint \vec{D}\cdot d\vec{a} = Q_{free} \Rightarrow D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

$$\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P} \Rightarrow D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}$$

"usual argument" whence $E^{\parallel}_{above} = E^{\parallel}_{below}$ previously



$$0 = \int \vec{\nabla}\times\vec{D}\cdot d\vec{a} = \int \vec{D}\cdot d\vec{l}$$

Consistent with earlier claim

$$E^{\perp}_{above} - E^{\perp}_{below} = \sigma/\epsilon_0 \qquad E^{\parallel}_{above} - E^{\parallel}_{below} = 0$$

Since
$$\underbrace{D^{\perp}_{above} - D^{\perp}_{below}}_{\sigma_f} = (\epsilon_0 E^{\perp}_{above} + P^{\perp}_{above}) - (\epsilon_0 E^{\perp}_{below} + P^{\perp}_{below})$$

$$= \underbrace{\epsilon_0 (E^{\perp}_{above} - E^{\perp}_{below})}_{\sigma/\epsilon_0} + \underbrace{P^{\perp}_{above} - P^{\perp}_{below}}_{-\sigma_b}$$

$\boxed{\text{Can use either formulation of bdy conditions, i.e. in terms of } \vec{D} \text{ or } \vec{E}}$

$$D^{\parallel}_{above} - D^{\parallel}_{below} = (\epsilon_0 E^{\parallel}_{above} + P^{\parallel}_{above}) - (\epsilon_0 E^{\parallel}_{below} + P^{\parallel}_{below})$$

$$= \epsilon_0 \underbrace{(E^{\parallel}_{above} - E^{\parallel}_{below})}_{0} + P^{\parallel}_{above} - P^{\parallel}_{below}$$

$P_{above} = 0$

$+ + +$

$P_{below} > 0$

EM 13

Linear dielectrics: Many materials obey

$$* \quad \vec{P} = \epsilon_0 \chi_e \vec{E}$$

"electric susceptibility"

IMPT: $\vec{E}$ in $*$ is total field due to external $\vec{E}_0$ and $\vec{P}$ itself!

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon_0 (1+\chi_e)\vec{E}$$

$$\vec{D} = \epsilon \vec{E}$$

$$\epsilon \equiv \epsilon_0 (1+\chi_e) \leftarrow \text{"permittivity"}$$

$$\epsilon_r \equiv \epsilon/\epsilon_0 = 1+\chi_e \leftarrow \text{"relative permittivity"}$$

or "dielectric constant"

$\epsilon_r = 1$ for free space (vacuum)

	1.000065	He
	1.000254	H_2
	1.000548	N_2
	1.00589	H_2O vapor
	80.1	H_2O liquid
	2.28	Benzene
	11.7	Si
What!? →	34000	$KTaNbO_3$

EM14

Metal sphere radius a charge Q surrounded by linear dielectric ϵ to radius b. Compute V at sphere center.

METHOD #1 :

$\vec{D} \to \vec{E}$ $\quad r > a \qquad \vec{D} = \frac{Q}{4\pi r^2}\hat{r}$ $\quad$ ($\vec{D}$ determined solely by free charge)

$$\vec{E} = \vec{D}/\epsilon = \begin{cases} \frac{Q}{4\pi r^2 \epsilon_0}\hat{r} & r > b \\ \frac{Q}{4\pi r^2 \epsilon}\hat{r} & a < r < b \end{cases}$$

METAL $\Rightarrow$ $\vec{E} = 0 \quad r < a$

$> b \qquad V(r) = -\int_\infty^r \vec{E}\cdot d\vec{l} = -\int_\infty^r \frac{Q}{4\pi\epsilon_0 r'^2}\hat{r}\cdot(\hat{r})dr'$

$$= \frac{Q}{4\pi\epsilon_0 r'}\Big|_\infty^r = \frac{Q}{4\pi\epsilon_0 r}$$

$a < r < b \qquad V(r) = \frac{Q}{4\pi\epsilon_0 b} - \int_b^r \frac{Q}{4\pi\epsilon r'^2} dr'$

$$= \frac{Q}{4\pi\epsilon_0 b} + \frac{Q}{4\pi\epsilon r'}\Big|_b^r$$

$$= \frac{Q}{4\pi}\left\{\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon r} - \frac{1}{\epsilon b}\right\}$$

V = constant inside so $V_{center} = V(a)$

$$= Q/4\pi \left\{ 1/\epsilon_0 b + 1/\epsilon a - 1/\epsilon b \right\}$$