EM15

METHOD #2

Q

Can also do by examining $\vec{P} \rightsquigarrow \sigma_b, \rho_b \rightarrow E$

$$\vec{P} = \epsilon_0 \chi_e \vec{E} = \epsilon_0 \chi_e \frac{Q}{4\pi r^2 \epsilon} \hat{n}$$

$$\rho_b = \vec{\nabla} \cdot \vec{P} = 0$$

$$\sigma_b = \vec{P} \cdot \hat{n} = \begin{cases} \epsilon_0 \chi_e \, Q/4\pi\epsilon b^2 & r = b \\ -\epsilon_0 \chi_e \, Q/4\pi\epsilon a^2 & r = a \end{cases}$$

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ $\qquad$ $\vec{E} = \frac{1}{\epsilon_0}(\vec{D} - \vec{P})$

Alternate derivation of $\sigma_b = \vec{P}\cdot\hat{n}$

We have $\vec{P} = \dfrac{\text{dipole moment}}{\text{volume}} = n\vec{p}$

individual dipole moment

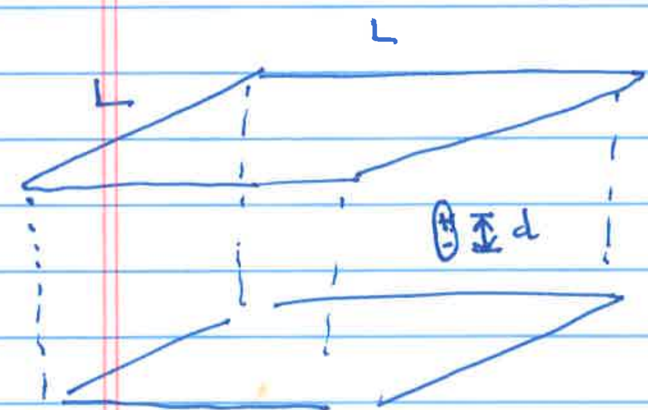
$n = \dfrac{\#\text{ dipoles}}{\text{volume}}$

First note units work out

$$[\vec{P}] = \frac{[p]}{L^3} = \frac{QL}{L^3} = \frac{Q}{L^2} = [\sigma]$$

(The units also work for volume charge density

$$[\vec{\nabla}\cdot\vec{P}] = \frac{1}{L}[\vec{P}] = \frac{Q}{L^3} = [\rho]$$
)



← how many dipoles adjacent to top surface?

$n L^2 d$

size of dipole

$\therefore$ Charge on top surface $Q = q n L^2 d$

$$\therefore \quad \sigma = \frac{Q}{L^2} = qnd = nqd = np = P$$

Some Examples

* Dielectric sphere in uniform $\vec{E}$ field

(we did metallic sphere in uniform $\vec{E}$ field!)

Same starting point

$r<R \qquad V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$

$r>R \qquad\qquad = \sum_{\ell=0}^{\infty} B_\ell r^{-(\ell+1)} P_\ell(\cos\theta) - E_0 r\cos\theta$

Different Boundary conditions

$V_{in} = V_{out} \quad @ \; r=R$

$\epsilon \frac{\partial V_{in}}{\partial r} = \epsilon_0 \frac{\partial V_{out}}{\partial r} \quad @ \; r=R$

$(D_\perp)_{above} - (D_\perp)_{below}$

$= \sigma_f = 0$

$(V_{out} \to -E_0 z \quad r \gg R)$

used already

$$\text{V cont}\quad \begin{cases} \sum A_\ell R^\ell P_\ell(\cos\theta) = -E_0 R\cos\theta + \sum \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) \\ A_\ell R^\ell = B_\ell / R^{\ell+1} \qquad \ell \neq 1 \\ A_1 R = -E_0 R + B_1/R^2 \qquad \ell = 1 \end{cases}$$

$$\text{D cont}\quad \begin{cases} \epsilon_r \sum \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) = -E_0\cos\theta - \sum \frac{(\ell+1)B_\ell}{R^{\ell+2}} P_\ell(\cos\theta) \\ \epsilon_r \ell A_\ell R^{\ell-1} = -(\ell+1)B_\ell / R^{\ell+2} \qquad \ell \neq 1 \\ \epsilon_r A_1 = -E_0 - 2B_1/R^3 \qquad \ell = 1 \end{cases}$$

$\Longrightarrow$ $A_\ell = B_\ell = 0$ for $\ell \neq 1$

$A_1 = -3/(\epsilon_r + 2)\, E_0$ $\qquad B_1 = \dfrac{\epsilon_r - 1}{\epsilon_r + 2} R^3 E_0$

$$V_{in}(r, \theta) = -\frac{3E_0}{\epsilon_r + 2} \underbrace{r\cos\theta}_{z}$$

$$\vec{E}_{in} = \frac{3}{\epsilon_r + 2} \underbrace{E_0 \hat{z}}_{\vec{E}_0}$$

} reduces to $\vec{E}_0$ for $\epsilon_r = 1$

W

also $B_1 \to 0$ no induced dipole

$\epsilon_r = 1$

$\sigma_b = \vec{P} \cdot \hat{n}$ or from $\frac{\partial E}{\partial r}$ discontinuity $= \sigma_b / \epsilon_0$

$\rho_b = -\vec{\nabla} \cdot \vec{P} = 0$ ← this is general if $\rho_f = 0$

(see next page)

In a linear dielectric, if there is no free charge, then there is no bound volume charge density either

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\nabla\cdot\left(\epsilon_0 \frac{\chi_e \vec{D}}{\epsilon}\right) = \frac{-\chi_e}{1+\chi_e}\underbrace{\vec{\nabla}\cdot\vec{D}}_{\rho_f}$$

$\Rightarrow$ all net charge must reside on surface.

Review: $\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E} = \epsilon_0(1+\chi_e)\vec{E} \equiv \epsilon\vec{E}$

$\vec{P}, \vec{D}, \vec{E}$ all multiples of each other

$$\vec{P} = \vec{D} - \epsilon_0\vec{E} = \vec{D} - \frac{\epsilon_0}{\epsilon}\vec{D} = \left(\frac{\epsilon-\epsilon_0}{\epsilon}\right)\vec{D}$$

$$\epsilon = \epsilon_0(1+\chi_e) \qquad \epsilon - \epsilon_0 = \epsilon_0\chi_e$$

$$\frac{\epsilon-\epsilon_0}{\epsilon} = \frac{\epsilon_0\chi_e}{\epsilon_0(1+\chi_e)} = \frac{\chi_e}{1+\chi_e}$$

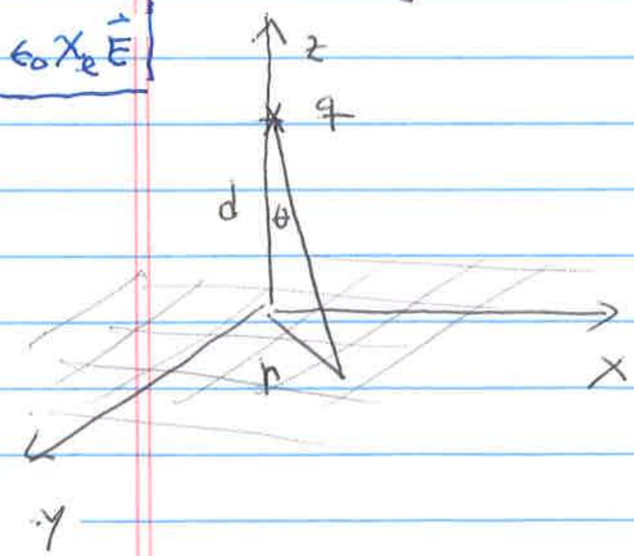
$\vec{D} = \epsilon\vec{E}$

$\vec{P} = \left(\frac{\epsilon-\epsilon_0}{\epsilon}\right)\vec{D}$

$= (\epsilon-\epsilon_0)\vec{E}$

$= \epsilon_0\chi_e\vec{E}$

Point charge above dielectric slab:



$\rho_b = 0$ by argument above

$$\sigma_b = \vec{P}\cdot\hat{n} = P_z = \epsilon_0\chi_e E_z$$

$$E_z^q = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2+d^2}\cos\theta$$

due to q

$$= -\frac{1}{4\pi\epsilon_0}\, qd/(r^2+d^2)^{3/2}$$

IMPT E_z is total field due to q and σ_b

$$E_z^b = -\sigma_b/2\epsilon_0$$

$$\sigma_b = \epsilon_0 \chi_e \left\{ \frac{-1}{4\pi\epsilon_0} \frac{qd}{(r^2+d^2)^{3/2}} - \frac{\sigma_b}{2\epsilon_0} \right\}$$

$$\sigma_b = -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right) qd/(r^2+d^2)^{3/2}$$

extra factor over result for conducting plane

previous page

$\vec{P} = \frac{\epsilon-\epsilon_0}{\epsilon} \vec{D}$

$\vec{P} = \vec{D}$

$\Rightarrow \vec{E} = 0$

$\Longrightarrow$ often conductor can be regarded as limiting case of dielectric with $\epsilon = \infty$

consistent with $\rho_b = 0$ inside...

$$q_b = -\chi_e/(\chi_e+2)\; q$$

skip — can get V by method of images

(i) Can never put image charge in region in which you are computing V

(ii) Image charges must add to correct total in each region

$z < 0$
$$V = \frac{1}{4\pi\epsilon_0} \, q+q_b/(x^2+y^2+(z-d)^2)^{1/2}$$

$z > 0$
$$V = \frac{1}{4\pi\epsilon_0}\left\{ \frac{q}{(x^2+y^2+(z-d)^2)^{1/2}} + \frac{q_b}{[x^2+y^2+(z+d)^2]^{1/2}} \right\}$$

can check $-\epsilon_0\left(\left.\frac{\partial V}{\partial z}\right|_{z=0^+} - \left.\frac{\partial V}{\partial z}\right|_{z=0^-} \right) = \sigma_b$

$\boxed{\text{Energy}}$

$$W = \frac{1}{2}\epsilon_0 \int E^2 \, d\tau \rightarrow \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau$$

See Griffiths 197-198 for derivation

See page 199 for discussion of difference between $\frac{1}{2}\epsilon_0 E^2$ and $\frac{1}{2}\,\vec{D}\cdot\vec{E}$. A subtle distinction not unlike the puzzle of why $\frac{1}{2}\epsilon_0 E^2$ always positive but $\frac{1}{4\pi\epsilon_0} q_i q_j / r$ can be negative.