

10/51/

In discussing de Moivre's Thm we used

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

This came from Taylor's theorem

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \dots$$

applied to $f(x) = e^x$ at $a=0$.

Similarly $\sin x = x - \frac{x^3}{3} + \frac{x^5}{120} - \dots$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$$

Other useful (commonly occurring) series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2} x^2 + \dots$$

works even if n is not an integer
(series terminates if n is an integer)

Series expansions are very useful when doing hard

problems where there is a small variation from a

solvble case.

NOTE:
infinite sum
of perfectly
well behaved
functions
(differentiable)
can result in
function with
singularities!

SIA

Comments.

(1) Although we will not discuss proofs of convergence

here, the practical issue of rate of convergence, how

very important
for computational
work!

many terms do you need to keep, is important.

$e^x, \sin x, \cos x$ converge fast because of factorials

in denominator

$$e^1 = 1 + 1 + \frac{1^2}{2} + \frac{1^3}{6} + \frac{1^4}{24} + \frac{1^5}{120} + \frac{1^6}{720} + \dots$$

$$= 2.71805 \text{ at } n=6 \quad ,00014$$

$$2.71828 \text{ exact}$$

There is no especially good series for π , interestingly!

[Q] Does anyone know?

$$\tan^{-1} x = \int_0^x \frac{du}{1+u^2}$$

$$\frac{\pi}{4} = \tan^{-1} 1 = \int_0^1 \frac{du}{1+u^2} = \int_0^1 du (1-u^2+u^4-u^6+\dots)$$

$$= u - u^3/3 + u^5/5 - u^7/7 + \dots - \int_0^1$$

$$= 1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11$$

SIB

$$\pi = 4 \left\{ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \dots \right\}$$

after 6 terms $\pi = 2.97605$

$$\pi = 3.14159$$

(2) More interesting: Taylor series tells you

special functions like $e^x, \sin x, \cos x$ can be

expanded in familiar set of polynomials. A natural

question is: Are there other sets of functions besides

polynomials which can serve as a "basis" for all

functions?



Q Does anyone know?

Fourier series
Legendre



Q What is basic principle behind such sets?



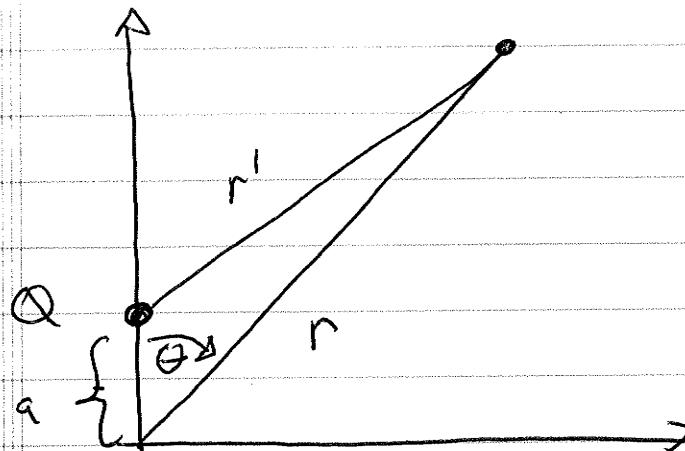
a really cool mix of physics and math!

We will answer this!

C₁, linear
algebra

S2//

Usefulness Potential due to charge Q
 at distance " a " from origin
 evaluated at point r, θ



"law of cosines"

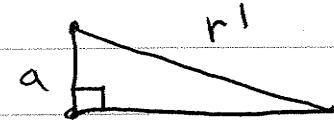
$$r'^2 = r^2 + a^2 - 2ar \cos \theta$$

$\theta = \pi/2$ is
 just pythagoras

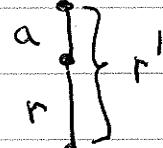
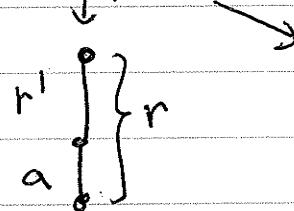
Setting $4\pi\epsilon_0 = 1$

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2$$

$$V = \frac{Q/r'}{4\pi\epsilon_0} \quad x = -\frac{2a}{r} \cos \theta + \frac{a^2}{r^2}$$



$\theta = 0$, r' are moral



$$r' = r \left(1 - \frac{2a}{r} \cos \theta + \frac{a^2}{r^2} \right)^{1/2}$$

$$= r \left(1 + \frac{a}{r} \cos \theta + \frac{a^2}{2r^2} + \frac{3}{8} \frac{4a^2}{r^2} \cos^2 \theta + \dots \right)$$

from

$$V = \frac{Q}{4\pi\epsilon_0 r} \left\{ 1 + \frac{a}{r} \cos \theta + \frac{a^2}{2r^2} (-1 + 3 \cos^2 \theta) + \dots \right\}$$

If you expand this \rightarrow expansion of V in powers of small quantity a/r
 Later we will encounter Legendre polynomials \leftarrow first hint
 Legendre polys useful in

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{-1}{2} + \frac{3x^2}{2}$$

~~$V(r, \theta) = \frac{Q}{r} \left\{ 1 + \dots \right\}$~~

$$V(r, \theta) = \frac{Q}{r} \sum_{n=0}^{\infty} P_n(\cos \theta) \left(\frac{a}{r} \right)^n$$

S2A

Notice $P_0(1) = P_1(1) = P_2(1) = 1$ This is true for all n .

Preview of next & last topic in course

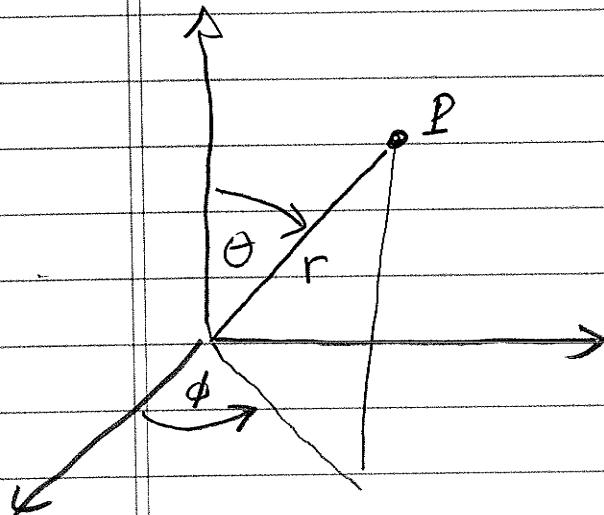
$$V(r) = \frac{Q}{r} \left\{ P_0(\cos\theta) + \frac{a}{r} P_1(\cos\theta) + \frac{a^2}{r^2} P_2(\cos\theta) + \dots \right\}$$

Any charge distribution which gives rise to $V(r)$ with

θ & ϕ dependence due to symmetry can be expanded

like this!

$$V(r) = \sum_{n=0}^{\infty} \frac{c_n P_n(\cos\theta)}{r^{n+1}}$$



All you need to do is

determine the coefficients c_n !

Trivial example: point charge Q

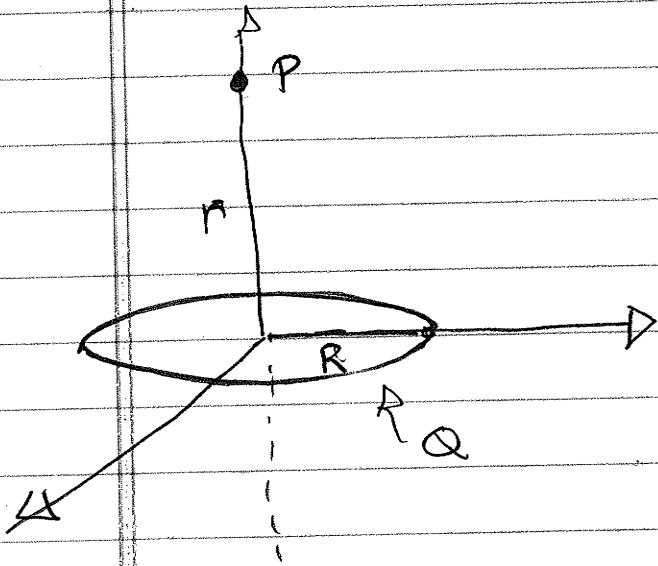
$$V(r) = \frac{Q}{4\pi\epsilon_0 r}$$

$$c_0 = \frac{Q}{4\pi\epsilon_0}$$

$$c_n = 0 \quad \text{all other } n.$$

S2B//

Potential due to ring of charge Q



Easy to get $V(r)$ along

the z -axis where $\theta = 0$

Since all points on

ring are same distance
 $\sqrt{r^2 + R^2}$ from P

$$V(\cos\theta=1) = \frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{r^2 + R^2}}$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{1}{r} (1 + R^2/r^2)^{-1/2}$$

$$= \frac{Q}{4\pi\epsilon_0 r} \left\{ 1 - \frac{R^2}{2r^2} + \left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right) \left(\frac{R^2}{r^2} \right)^2 - \dots \right\}$$

$$= \frac{Q}{4\pi\epsilon_0 r} \left\{ 1 - \frac{R^2}{2r^2} + \frac{3}{8} \frac{R^4}{r^4} - \dots \right\}$$

Compare $\left| \begin{array}{l} = c_0 \frac{P_0(l)}{r} + \frac{c_1 P_1(l)}{r^2} + \frac{c_2 P_2(l)}{r^3} + \frac{c_3 P_3(l)}{r^4} + \dots \end{array} \right.$

$$\text{use } P_n(l) = 1$$

$$c_0 = \frac{Q}{4\pi\epsilon_0} \quad c_2 = -\frac{Q}{4\pi\epsilon_0} \frac{R^2}{2} \quad c_4 = \frac{Q}{4\pi\epsilon_0} \left(\frac{3R^4}{8} \right)$$

$$c_1 = c_3 = c_5 = \dots = 0$$

S2C

From this we learn potential at all points

in space even away from z axis ! $P_2(\cos\theta)$

$$V(r, \theta) = \frac{\Omega}{4\pi G_0} \frac{1}{r} \left\{ 1 - \frac{1}{2} \frac{R^2}{r^2} \left(-\frac{1}{2} + \frac{3 \cos^2\theta}{2} \right) \right.$$

no ϕ dependence \nearrow

$$\left. + \frac{3}{8} \frac{R^4}{r^4} \left(\frac{35 \cos^4\theta - 30 \cos^2\theta + 3}{8} \right) \right.$$

$P_4(\cos\theta)$

53

Differentiation trick of §3 is often useful for series.

First note that it works for common Taylor series

$$e^x = 1 + x + x^2/2 + x^3/6 + x^4/24 + \dots$$

$$\frac{d}{dx} e^x = 1 + x + x^2/2 + x^3/6 + \dots$$

$$\text{or } \cos x = 1 - x^2/2 + x^4/24 - x^6/720$$

$$\frac{d}{dx} \sin x = -x + x^3/6 - x^5/120 \quad \checkmark$$

$$\text{But others } \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$$

$$\frac{d}{dx} \left(\frac{1}{(1-x)^2} \right) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$$

Thus, for example if $x = 1/2$

$$1 + 1 + 3/4 + 4/8 + 5/16$$

$$= 4 = \frac{1}{(1-1/2)^2}$$

T1

Solving transcendental eqns

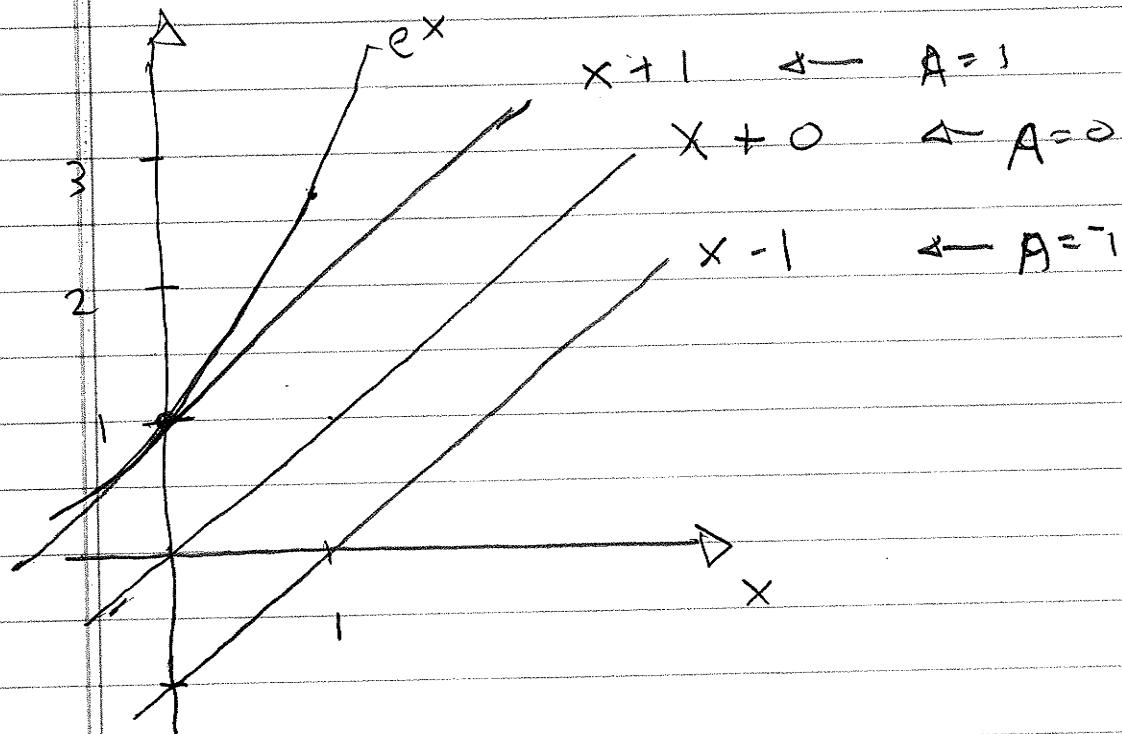
$$e^x = x + A$$

Q: For what values of A does one have a solution

and how does the solution x behave near the value of A

where solutions first appear?

A picture is always useful



(Clearly get soln only for $A \geq 1$)

and soln is $x=0$ at $A=1$

\uparrow
"critical value of A "
 $\equiv A_c$

T2

To get value of x for A just a bit bigger

than $A=1$, expand e^x

$$1 + x + \frac{x^2}{2} = x + A$$

$$x + \frac{x^2}{2} = x + A - 1$$

$$x^2 = 2(A-1)$$

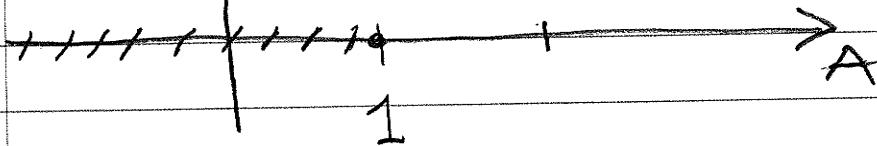
$$x = \pm \sqrt{2(A-1)}$$

x

$$x = \pm \sqrt{2(A-A_0)}$$

A	x approx	x exact
1.01	.1414	.1382
1.05	.3162	.3004
1.1	.4472	.4162

no solns



P102

assignment
#2!

Bi

Binomial series

$$(p+q)^N = \sum_{n=0}^N \binom{N}{n} p^n q^{N-n}$$

$$\frac{N!}{n!(N-n)!}$$

e.g H and T
in coin toss

Most important application N independent events w/ 2 outcomes

each event prob p for H and $q = 1-p$ for T

probability of n H and $N-n$ T is $\binom{N}{n} p^n q^{N-n}$

Examples $N=2$

outcome prob

HH p^2 → prob $n=2$ H is $p^2 = \binom{2}{2} p^2 q^0$

HT pq } $n=1$ H is $2pq = \binom{2}{1} p^1 q^1$

TH qp

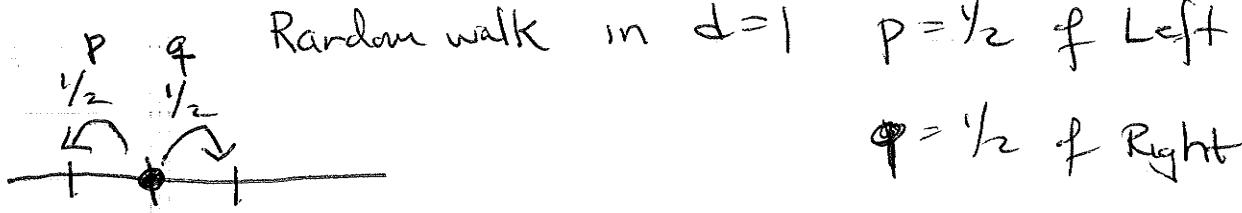
TT q^2 → prob $n=0$ H is $q^2 = \binom{2}{0} p^0 q^2$

sum of prob of all outcomes

$$p^2 + 2pq + q^2 = (p+q)^2 = 1$$

and also generally for $N > 2$

B2



$$\text{prob of } n \text{ Left} = \binom{N}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{N-n} = \binom{N}{n} \left(\frac{1}{2}\right)^N$$

Q What is average final location if starting at origin

Answer is obviously ϕ but to prove

$$\text{final location} = \underbrace{N - n}_{\substack{\text{steps} \\ \text{to Right}}} - \underbrace{n}_{\substack{\text{steps to left} \\ +1}} = N - 2n$$

$$\langle x_f \rangle = \sum_{n=0}^N \underbrace{\binom{N}{n} \left(\frac{1}{2}\right)^N}_{\substack{\text{all possible} \\ \text{outcomes}}} (N - 2n) \uparrow \begin{matrix} \text{prob of} \\ \text{particular} \\ \text{outcome} \end{matrix} \quad \begin{matrix} \text{final location} \end{matrix}$$

$$= N - 2 \sum_{n=0}^N \binom{N}{n} \left(\frac{1}{2}\right)^N n$$

Q what is your guess for $\langle x_f \rangle$ in general case where $p \neq q$

Answer: $\langle x_f \rangle = N(q-p)$

B3

Useful trick is differentiating series

$$\sum_{n=0}^N \binom{N}{n} p^n q^{N-n} = (p+q)^N$$

$$\frac{\partial}{\partial p} \sum_{n=0}^N \binom{N}{n} np^{n-1} q^{N-n} = N(p+q)^{N-1}$$

set $p=q=\frac{1}{2}$

$$\sum_{n=0}^N \binom{N}{n} n \left(\frac{1}{2}\right)^{N-1} = N$$

- Or - for any
 $p+q=1$ get

$$\sum \binom{N}{n} np^{n-1} q^{N-n} = N$$

$\checkmark \times$ sum on page B2

$$\rightarrow \sum \binom{N}{n} np^n q^{N-n} = Np$$

$$\langle x_f \rangle = N - 2 \frac{1}{2} N = 0 \quad \leftarrow p=q=\frac{1}{2}$$

$$\rightarrow \langle x_f \rangle = N - 2Np = N(1-2p) = N(p+q-2p) = N(q-p)$$

A nontrivial and incredibly important result is

$$\langle x_f^2 \rangle = N$$

$$\sqrt{\langle x_f^2 \rangle} = \sqrt{N} \quad \leftarrow \text{rms distance from origin is proportional to square root of the steps}$$

Distance particle travels in diffusion

$$\sim \sqrt{t} \propto \text{time}$$

B4

HW problem to prove this!

$$\langle x_f \rangle^2 = \sum_{n=0}^N (N-2n)^2 \binom{N}{n} \left(\frac{1}{2}\right)^N$$

$$= \sum_{n=0}^N (N^2 - 4nN + 4n^2) \binom{N}{n} \left(\frac{1}{2}\right)^N$$

$$= N^2 - 4N \sum_{n=0}^N n^2 + 4 \sum_{n=0}^N n^2 \binom{N}{n} \left(\frac{1}{2}\right)^N$$

↓
did this one
before

$$\sum_{n=0}^N \binom{N}{n} np^{n-1} q^{N-n} = N(p+q)^{N-1}$$

xp

$$\sum_{n=0}^N \binom{N}{n} np^n q^{N-n} = Np(p+q)^{N-1}$$

$\frac{\partial}{\partial p}$ again

$$\sum \binom{N}{n} n^2 p^{n-1} q^{N-n} = N(p+q)^{N-1} + N(N-1)p(p+q)^{N-2}$$

$$\sum n^2 \binom{N}{n} \left(\frac{1}{2}\right)^{N-1} = N + N(N-1)/2$$

$$\sum n^2 \binom{N}{n} \left(\frac{1}{2}\right)^N = \frac{N}{2} + \frac{N^2}{4} - \frac{N}{4}$$

$$\langle x_f^2 \rangle = N^2 - 2N^2 + 4 \left(\frac{N}{4} + \frac{N^2}{4} \right) = N \quad \blacksquare$$

PT1

Phase Transitions and Series Expansions

The theory of phase transitions is an example where expansions in a small quantity is a very useful tool. There are actually several expansions one can do. Prof Singh's perhaps the world expert in expansions in high temperature (where $1/T$ is the small expansion).

What are phase transitions? Abrupt change in properties of material. You know

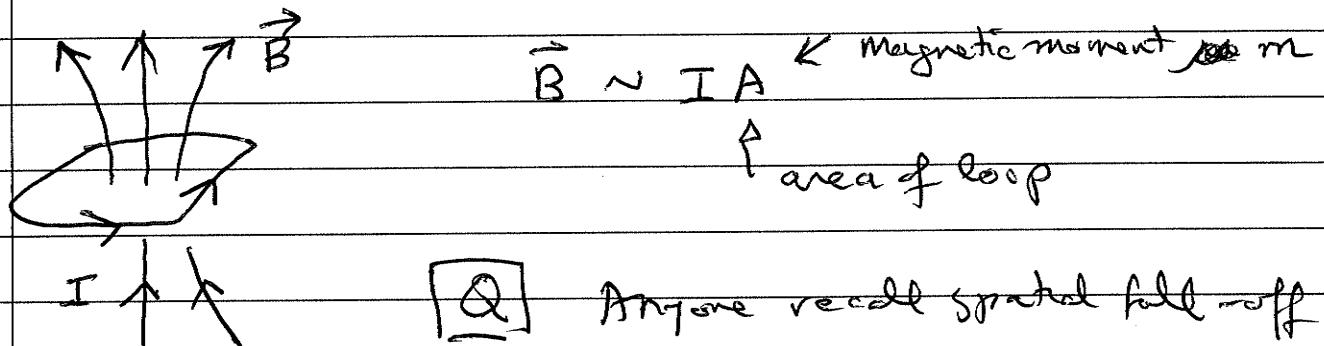
example: gas-liquid-solid eg H₂O

Surprise is the abruptness. Why do properties not change continuously?

PT 2

Consider magnetic phase transitions. Recall

Physics 9C. Current loops produce magnetic field



Anyone recall spatial fall-off
with r ? $\vec{B} \sim 1/r^3$

Compare to \vec{E} from point charge

$$\vec{E} \sim 1/r^2$$

There are no magnetic

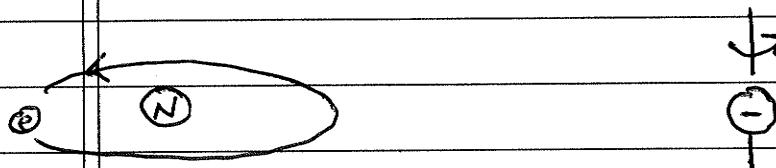
monopoles. Always
have $[N \ S]$ pairs

This r dependence is more like
that of a dipole $(\oplus \ominus)$

and $1/r^3$ fall off. R. Dirac 1931 paper, some really cool stuff!

If there were monopoles could prove charge
is quantized!

Many such "current loops" in a solid.



electron orbiting
nucleus

electron
spinning around
axis.

In most solids the \vec{B} fields are randomly oriented
because orbits/spins are in many different directions
 \rightarrow cancel \rightarrow non magnetic.

PT3

Current loops \Rightarrow get preferred direction

if you apply an external field, \vec{B}_{ext}

$$\mathbf{E} = -\vec{m} \cdot \vec{B}_{\text{ext}}$$

\vec{m} lines up \parallel to \vec{B}_{ext}

\vec{B}_{ext} "breaks rotational symmetry" and picks out

a preferred direction.

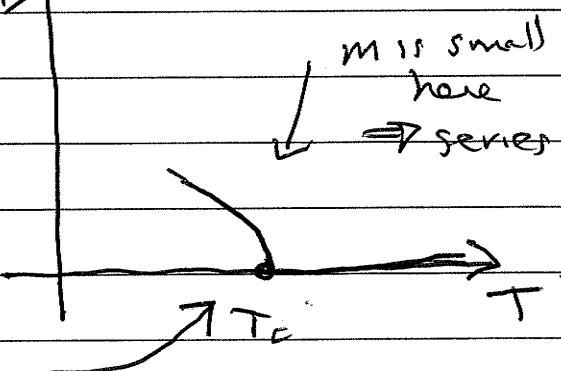
$$\langle \vec{m} \rangle = \chi \vec{B}_{\text{ext}}$$

["magnetic susceptibility"]

In phase transition $\langle \vec{m} \rangle$ becomes non-zero

Even if $\vec{B}_{\text{ext}} = 0$ "spontaneous symmetry breaking"

This is really weird! $\langle m \rangle \uparrow$



Your HW job is to figure out functional form of $\langle m \rangle(T)$

$$\text{from } m = \tanh \beta J/T$$

PT2A

Skip?

Landau Theory

Expand (free) energy in powers of order parameter m ; assume symmetry rules out odd powers

$$f(m) = a(T)m^2 + b(T)m^4$$

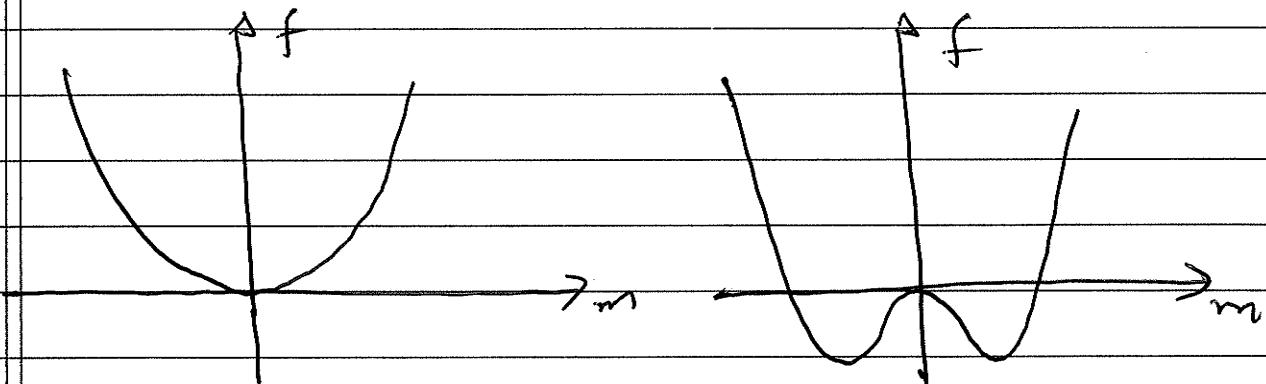
At phase transition $a(T)$ changes sign

$$a(T) < 0 \quad T < T_c$$

$$a(T) > 0 \quad T > T_c$$

for $T > T_c$

$T < T_c$

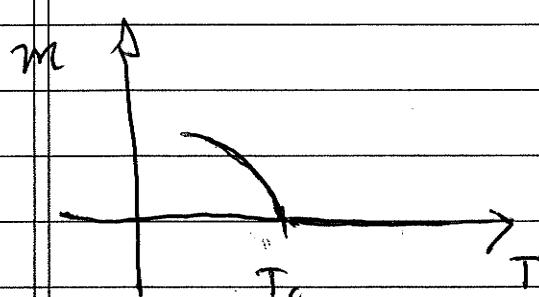


$m = 0$ minimizes

$m \neq 0$ minimizes

$$f'(m) = 2am + 4m^3b$$

$$m=0 \text{ and } m = \sqrt{-a/2b}$$



recall $a < 0$ for $T < T_c$

PT3B

Usual (simplest) assumption is that

$$\alpha(T) = c(T - T_c)$$

↑
Some constant

so that $m = \sqrt{\frac{c}{2b}} (T_c - T)^{\frac{1}{2}}$

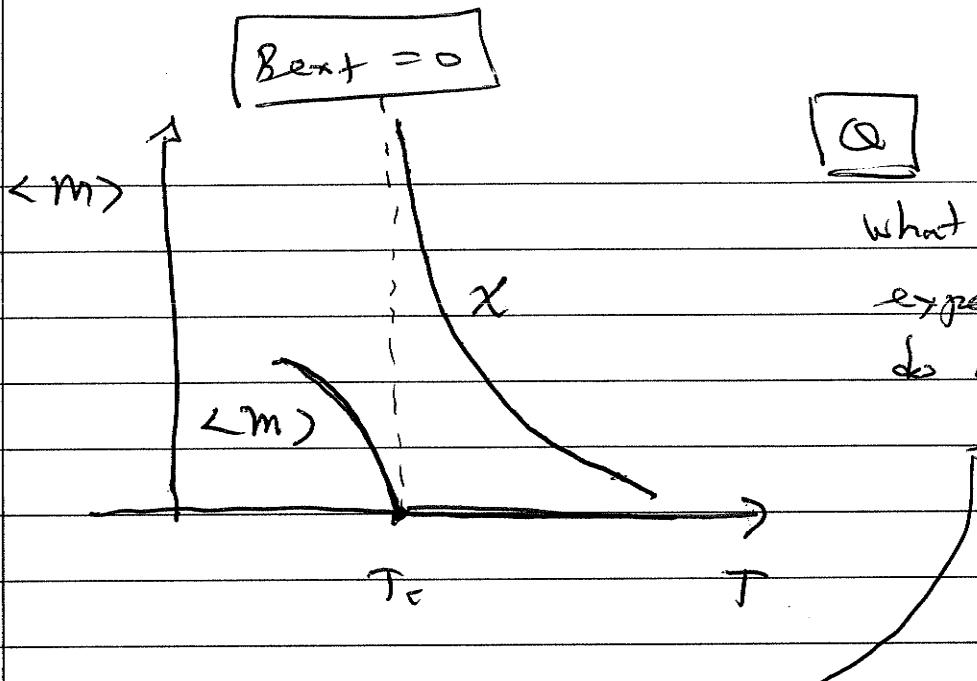
for $T < T_c$

Compare this to what you find

in your problem set when you solve

$$m = \tanh Jm/T$$

PT4



What do we
expect χ to
do at T_c ?

If you have a material which is just
about to order on its own without any B_{ext}
Do you expect its susceptibility to be
large or small?

$$\chi \sim \text{diverges at } T_c \sim \frac{1}{(T - T_c)^\beta} \quad \beta$$

critical exponent β

I will evaluate β

for you. Your job is to get β : $m \sim (T_c - T)^\beta$

PT5

$$\text{In HW} \quad m = \tanh \frac{mJ}{T}$$

This is for $B_{ext} = 0$. More generally

$$m = \tanh \left(\frac{mJ + B_{ext}}{T} \right)$$

Differentiate wrt B $x = dm/dB$

$$x = \operatorname{sech}^2 \left(\frac{mJ + B_{ext}}{T} \right) \left[x \frac{J}{T} + 1 \right]$$

1) Set $B_{ext} = 0$

2) In HW see $T_c = J$

3) m is small, what is sech^2

\curvearrowleft series enter $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

$$\operatorname{sech}^2 x = \frac{2}{(1+x+\frac{x^2}{2}+\dots)(1-x+\frac{x^2}{2}\dots)} = \frac{1}{1+x^2+\dots}$$

$$= 1 - x^2 + \dots$$

$$\cos x = 1 - x^2/2 + x^4/24 - \dots$$

$$\cosh x = 1 + x^2/2 + x^4/24 - \dots = \cos(ix)$$

PT6

Actually, all we need is very first term $\propto e^{k(T-T_c)}$

$$x = x \left(\frac{T_c}{T} + 1 \right)$$

$$x = \frac{1}{1 - T_c/T}$$

\curvearrowleft diverges as $T \rightarrow T_c$ as expected.

skip?

Possibly silly analogy

$$\begin{bmatrix} \text{Your progress} \\ \text{on HW} \end{bmatrix} = x \begin{bmatrix} \text{Your friend suggests} \\ \text{you do your HW} \end{bmatrix}$$

\uparrow
HW susceptibility

