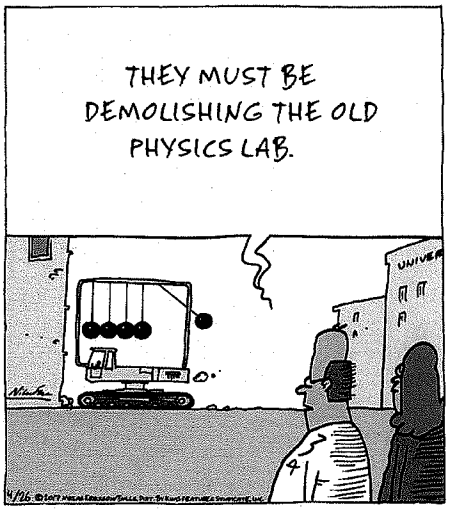


MIDTERM EXAM

PHYSICS 104A, FALL 2018
MATHEMATICAL PHYSICS

Please provide complete solutions which show all the work from which your answers come.

[1.] Compute $\ln(24 - 7i)$.

[2.] (a) Compute the eigenvalues and eigenvectors of the matrix,

$$M = \begin{pmatrix} 8 & 12 & 0 \\ 12 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

What relation do the eigenvectors have relative to each other? Does the form of M have anything to do with your answer?

[3.] Consider the force

$$\vec{F} = 4y\,\hat{\imath} - 2x\,\hat{\jmath}$$

Compute the integral of $\vec{F}$ from $(0,0)$ to $(1,2)$
(a) along the path $y = 2x$; and
(b) along the path $y = 2x^2$.
Describe any relevant general theorem which provides insight into your results.

[4.] Compute

$$\int_0^{2\pi} \frac{d\,\theta}{5 + 4\cos\theta}$$

[5.] (a) State De-Moivre's theorem.
(b) What is the basic idea behind proving it?
(c) Use De-Moivre's theorem to prove $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.

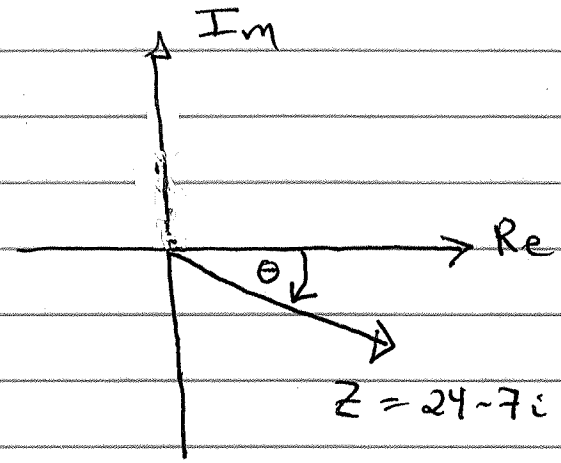
[6.] You were listening to the election news on Wednesday, and the television announcer asserted that the entire population of Florida had promised to vote completely randomly, with probability $p = 1/2$ for Democrat Andrew Gillum and $q = 1/2$ for Republican Ron DeSantis. The final vote was $2,100,000$ to $1,900,000$. How likely is it the people of Florida kept their promise? Explain. (I expecting just 2-3 sentences here which enunciate an important general principle and then show how it applies to this situation.)

Physics 104A
Midterm Exam Solutions

$\boxed{1}$ $24 - 7i = re^{i\theta}$ where

$r = \sqrt{7^2 + 24^2} = 25$

$\theta = \tan^{-1}(-7/24) = -0.284$ radians



$\ln(re^{i\theta}) = \ln r + i\theta$

So $\ln(24 - 7i) = \ln 25 - i\,0.284$

$= 3.219 - i\,0.284$

Could check by exponentiating eg

$e^{3.219} = 25$

$e^{-i\,0.284} = \cos(0.284) - i\sin(0.284)$

$= 0.96 - i(0.28)$

$e^{3.219 - i\,0.284} = 25(0.96 - i\,0.28)$

$= 24 - 7i$ ✓✓

2-1

Eigenvalues are determined by $|M-\lambda I|=0$

$$\begin{vmatrix} 8-\lambda & 12 & 0 \\ 12 & 1-\lambda & 0 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)\left[(8-\lambda)(1-\lambda)-144\right]=0$$

$$(7-\lambda)\left[\lambda^2-9\lambda-136\right]=0$$

$$(\lambda-17)(\lambda+8)$$

Eigenvalues $\lambda = 7, 17, -8$

For $\lambda=7$ eigenvector $v=\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ must obey

$$\begin{pmatrix} 1 & 12 & 0 \\ 12 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This eqn says nothing about c but gives $a=b=0$ because we have two homogeneous eqns in two unknowns which are linearly independent:

$$\begin{aligned} a+12b &= 0 \\ 12a-6b &= 0 \end{aligned} \Rightarrow a=b=0$$

So the eigenvector for $\lambda=7$ is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ normalized to length 1

$\vec{v}_1$ //

2-2

For $\lambda = 17$: $\begin{pmatrix} -9 & 12 & 0 \\ 12 & -16 & 0 \\ 0 & 0 & -10 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

We immediately see $c = 0$. The eqns for a and b are <u>dependent</u>

$$3\,(-3a + 4b) = 0 \quad \leftarrow \text{first eqn}$$
$$-4\,(-3a + 4b) = 0 \quad \leftarrow \text{second eqn}$$

So we have $-3a + 4b = 0 \Rightarrow \frac{1}{5}\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} = \vec{V}_2$

is (normalized) eigenvector

For $\lambda = -8$ $\begin{pmatrix} 16 & 12 & 0 \\ 12 & 9 & 0 \\ 0 & 0 & 15 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

again $c = 0$ and <u>dependent</u> eqns for a, b

$$4a + 3b = 0 \Rightarrow \frac{1}{5}\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} = \vec{V}_3 \quad \text{is eigenvector}$$

The eigenvectors are perpendicular to each other. Eg.

$\vec{V}_1 \cdot \vec{V}_2 = (0\ 0\ 1)\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} = 0 \qquad \vec{V}_2 \cdot \vec{V}_3 = (4\ 3\ 0)\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} = 0$

This $\boxed{\text{must}}$ occur because M is Hermitian (real symmetric)

3-1

$$\vec{F} = 4y\,\hat{i} - 2x\,\hat{j}$$

a) $y = 2x$

$dy = 2dx$

$$\int \vec{F}\cdot d\vec{r} = \int F_x\,dx + F_y\,dy$$

$$= \int_0^1 4\underbrace{(2x)}_{y}\,dx - 2x\underbrace{(2dx)}_{dy}$$

$$= \int_0^1 4x\,dx = 2x^2\Big|_0^1 = 2$$ (Joules if $\vec{F}$ in Newtons and x, y in meters)

b) $y = 2x^2$

$dy = 4x\,dx$

$$\int \vec{F}\cdot d\vec{r} = \int_0^1 4\underbrace{(2x^2)}_{y}\,dx - 2x\underbrace{(4x\,dx)}_{dy}$$

$$= \int_0^1 0\,dx = 0.$$

The works along the two paths are <u>not</u> <u>equal</u>.

That's okay because $\vec{F}$ is not conservative

$$\partial F_y/\partial x = -2 \qquad \partial F_x/\partial y = 4$$

It is interesting to ask why $W=0$ along path b.

$$\vec{F} = 4y\hat{i} - 2x\hat{j} = 8x^2\hat{i} - 2x\hat{j}$$

$$= 2x(4x\hat{i} - \hat{j})$$

and $$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$= dx(\hat{i} + 4x\hat{j})$$

You can see that $\vec{F}$ and $d\vec{r}$ are always $\perp$ to each other along path b! Hence $W=0$.

4-1

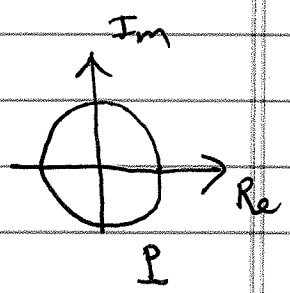
Writing $z = e^{i\theta}$ we have

$$\cos\theta = (e^{i\theta} + e^{-i\theta})/2 \qquad \text{and } dz = ie^{i\theta}d\theta$$

$$= (z + 1/z)\tfrac{1}{2} \qquad d\theta = \frac{dz}{iz}$$

Also as θ goes from $\emptyset$ to 2π z follows

a circular path P around the origin



$$I = \int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \oint_P \frac{dz}{iz} \cdot \frac{1}{5+2(z+1/z)}$$

$$= \oint_P \frac{1}{i} \frac{dz}{2z^2+5z+2}$$

$$z = \frac{-5 \pm \sqrt{5^2 - 4(2)(2)}}{2(2)} = \frac{-5 \pm \sqrt{9}}{4}$$

$$z = -2 \qquad z = -1/2$$

Can see this also by factorizing:

$$2z^2+5z+2 = (2z+1)(z+2)$$

The pole at $z = -1/2$ is inside the circle and

has residue $\quad R = \lim_{z \to -1/2} (z+1/2) \dfrac{1}{(2z+1)(z+2)}$

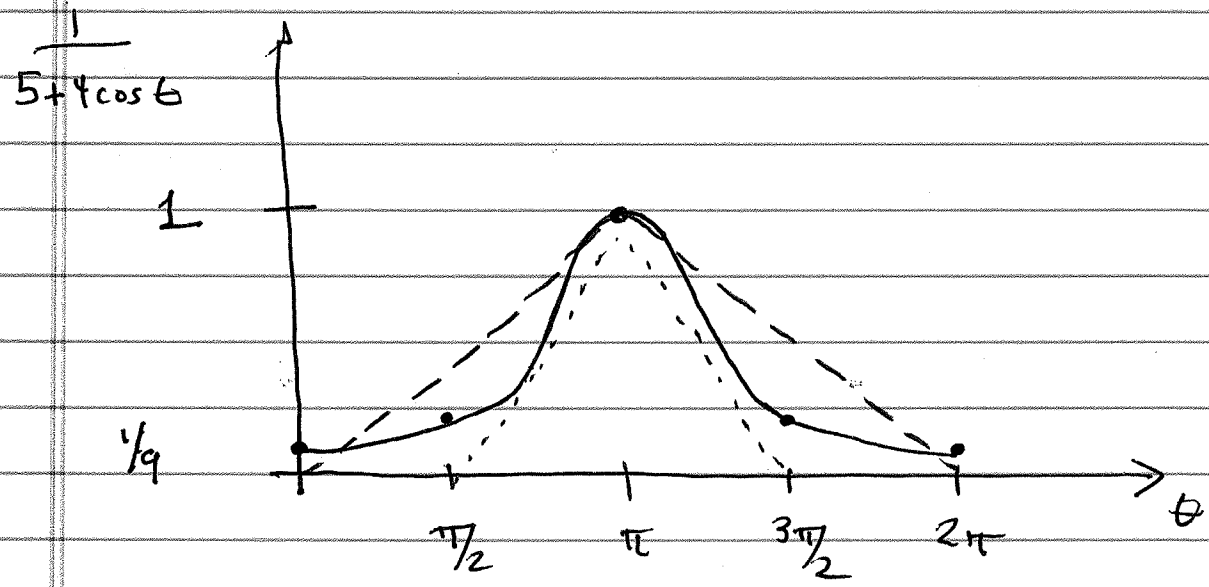
4-2

$$R = \lim_{z \to -1/2} \frac{1}{2} \frac{1}{z+2} = 1/3$$

So from Residue theorem

$$I = 1/i \; 2\pi i \left(1/3\right) = 2\pi/3.$$

Check this pictorially:



Seems like $I <$ area under ---

$$I < \frac{1}{2}(2\pi)1 = \pi$$

But probably bigger than area under ....

$$I > \frac{1}{2}\pi 1 = \pi/2.$$

So $2\pi/3$ is reasonable

5-1

(a) $e^{i\theta} = \cos\theta + i\sin\theta$

(b) <u>proof</u> comes from using Taylor series for sin, cos, and exponential. Actually doing it:

$$e^{x} = 1 + x + x^2/2 + x^3/6 + \cdots$$

$$e^{i\theta} = 1 + i\theta - \theta^2/2 - i\theta^3/6$$

$$= \underbrace{\left(1 - \theta^2/2 + \ldots\right)}_{\cos\theta} + i\underbrace{\left(\theta - \theta^3/6 + \ldots\right)}_{\sin\theta}$$

(c) $e^{2i\theta} = \cos 2\theta + i\sin 2\theta$

$$e^{2i\theta} = (e^{i\theta})(e^{i\theta}) = (\cos\theta + i\sin\theta)^2$$

$$= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$$

So $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

In a random walk of N steps with $p=q=\frac{1}{2}$

The most likely outcome is to end where you begin (at the origin). We know that other outcomes are possible, but are reasonably likely only if $\sqrt{\langle x^2 \rangle} \sim \sqrt{N}$.

Applying that here $N = 4{,}000{,}000$ and $\sqrt{N} = 2000$

We'd expect

$$\left.\begin{aligned} \text{Gillum} &= 2{,}000{,}000 + 2000 \\ \text{DeSantis} &= 2{,}000{,}000 - 2000 \end{aligned}\right\} \text{2000 vote difference is "expected"}$$

but a 100,000 vote difference is just impossible!

not expected on exam/solutions!

You could actually work it out with Stirling's formula $\ln N! = N \ln N - N + \frac{1}{2} \ln 2\pi N$

deviation	ratio of prob to most likely outcome (tie)	
100	.9950	
200	.9802	
500	.8825	
1000	.6065	
2000	.1353	
5000	.0000	$\leftarrow e^{-12.5}$
10000	.0000	$\leftarrow e^{-50}$
20000	.0000	$\approx e^{-200}$