

PHYSICS 104A, FALL 2016  
MATHEMATICAL PHYSICS  
Midterm Exam

[1.] Compute the following integral by the method of complex integration:

$$\int_0^{+\infty} \frac{1}{x^2 + 25} dx .$$

Explain all your steps clearly, e.g. what path you are using in the complex plane, how you arrive at this path, etc.

[2.] (a) Compute the eigenvalues and eigenvectors of the matrix,

$$M = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$$

What relation do the eigenvectors have relative to each other? Does the form of  $M$  have anything to do with your answer?

(b) If  $M$  is a quantum mechanical observable and  $\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is a particle's wave function, what are the possible values if you measure  $M$ , and what are their probabilities?  
(c) Compute the eigenvalues **only** of the matrix,

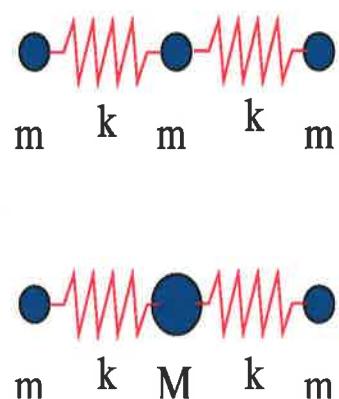
$$S = \begin{pmatrix} 6 & 5/2 \\ -5/2 & 2 \end{pmatrix}$$

Are the eigenvalues real? Could  $S$  be the matrix for a QM observable?

[3.] A particle undergoes a random walk of five steps, with probability  $p = 0.7$  of  $\Delta x = +1$  (i.e. to the right), and probability  $q = 0.3$  of  $\Delta x = -1$  (i.e. to the left). What are the possible final positions? What are their probabilities?

[4.] Your teaching assistant Alex discovers a new class of "Giguere operators"  $\mathcal{G}$ . They obey the relation  $\mathcal{G} = \mathcal{G}^3$ . What can you say about their eigenvalues?

[5.] In class we solved the problem of three equal masses  $m$  connected by two equal springs  $k$ . See figure. The frequencies were  $m\omega^2 = 0$ ,  $m\omega^2 = k$ , and  $m\omega^2 = 3k$ . The corresponding eigenvectors were  $(1 \ 1 \ 1)$ ,  $(1 \ 0 \ -1)$ ,  $(1 \ -2 \ 1)$ . Suppose we make the central mass larger  $M = 3m$  than the two end masses  $m$ . Without doing the problem mathematically, what can you say about the two lowest frequencies and associated normal mode vectors? To receive credit, you must clearly explain *why* you are making your various statements.



Physics 104 A

Midterm Exam

Fall 2016

$$\boxed{1} \quad \int_0^\infty \frac{dx}{x^2 + 25} = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{x^2 + 25}$$

because the integrand is an even function of  $x$ .

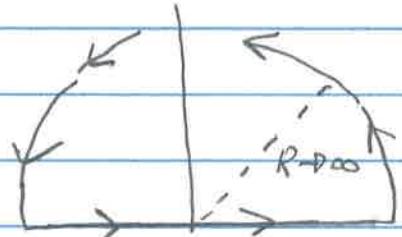
$$\int_{-\infty}^\infty \frac{dx}{x^2 + 25} = \int_R \frac{dz}{z^2 + 25}$$

where  $R$  is the path in the complex plane along the real axis, since  $z = x$  there. Finally,

$$\int_R \frac{dz}{z^2 + 25} = \oint_C \frac{dz}{z^2 + 25}$$

The contour  $C$ :

where  $C$  is a closed contour along real axis and a semicircle in the upper half plane. This is so because  $\frac{1}{z^2 + 25} \sim \frac{1}{R^2}$  on the semicircle and even though the semicircle's length  $\sim R$ ,  $R^{-1/R^2} \sim R^{-1}$  vanishes as  $R \rightarrow \infty$



$$\text{Now } \frac{1}{z^2 + 25} = \frac{1}{(z+5i)(z-5i)}$$

and hence the residue is

$$\lim_{z \rightarrow 5i} \frac{(z-5i)}{z^2 + 25} = \frac{1}{z+5i} \Big|_{z=5i} = \frac{1}{10i}$$

2.

So the integral is

$$\frac{1}{2} 2\pi i \frac{1}{10i} = \frac{\pi}{10}.$$

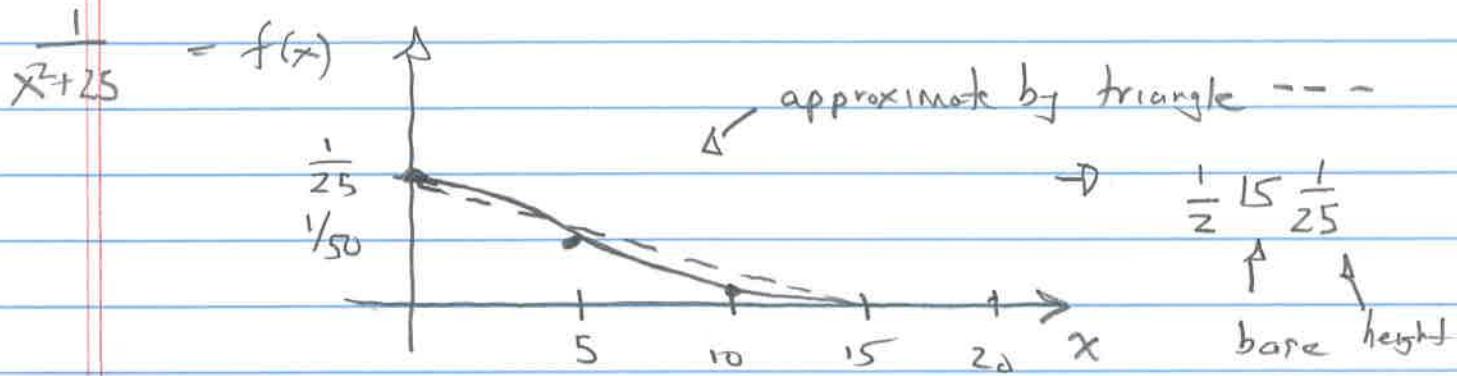
Check : Do by standard method of trig substitution.

$$\int_0^\infty \frac{dx}{x^2+25} = \int_0^{\pi/2} \frac{5 \sec^2 \theta d\theta}{25(1+\tan^2 \theta)} = \frac{1}{5} \int_0^{\pi/2} d\theta$$

$$x = 5 \tan \theta \quad \sec^2 \theta = \pi/10 \quad \checkmark$$

$$dx = 5 \sec^2 \theta d\theta$$

Another check is to estimate the area:



integral  $\sim \frac{3}{10}$  which is close to  $\frac{\pi}{10}$ .

Note: This sort of estimate is probably good enough to catch factor of 2 errors!

3.

$$\begin{aligned}
 \boxed{2} \quad (4) \begin{vmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} &= (6-\lambda)(3-\lambda) - 4 \\
 &= 18 - 9\lambda + \lambda^2 - 4 \\
 &= \lambda^2 - 9\lambda + 14 = (\lambda-2)(\lambda-7)
 \end{aligned}$$

Eigenvalues  $\lambda = 2, \lambda = 7$

$\lambda = 2$  eigenvector:  $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

obviously linearly dependent

$$2a + b = 0 \quad b = -2a \quad \vec{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

normalize

check:  $\begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda = 7$  eigenvector  $\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$-a + 2b = 0 \quad \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Obviously  $\vec{v}_1 \cdot \vec{v}_2 = 0$ .  $M$  is symmetric

so we expected eigenvectors to be perpendicular

4.

The possible values for measuring  $M$  are the eigenvalues  $\lambda = 2, 7$ .

To get probabilities  $\vec{\psi} = q_1 \vec{v}_1 + q_2 \vec{v}_2$

$$\text{Since } \vec{v}_i \cdot \vec{v}_j = \delta_{ij} \quad \vec{v}_1 \cdot \vec{\psi} = q_1, \quad \vec{v}_2 \cdot \vec{\psi} = q_2$$

$$p_1 = |q_1|^2 = \left| \frac{1}{\sqrt{5}} (1 - 2) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{10}} (-1) \right|^2 = 1/10$$

$$p_2 = |q_2|^2 = \left| \frac{1}{\sqrt{5}} (2 - 1) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{10}} 3 \right|^2 = 9/10.$$

$$(b) \quad \begin{vmatrix} 6-\lambda & 5/2 \\ -5/2 & 2-\lambda \end{vmatrix} = (6-\lambda)(2-\lambda) + 25/4$$

$$= \lambda^2 - 8\lambda + 12 + 25/4$$

$$= \lambda^2 - 8\lambda + 73/4$$

$$\lambda = \frac{1}{2} \left\{ 8 \pm \sqrt{64 - 73} \right\} = \frac{1}{2} \left\{ 8 \pm \sqrt{-9} \right\}$$

$$\lambda = 8/4 \pm 3/2 i$$

Eigenvalues are not real. (This is okay because  $S'$  is not symmetric.)  $S$  cannot be the matrix for a QM observable.

5.

3. Possible final positions are

- 5      ← 5 steps to left
- 3      ← 4 steps to left, 1 to right
- 1      ← 3 steps to left, 2 to right
- +1      :
- +3      :
- +5      ← 5 steps to right

To get their probabilities, use Pascal triangle

1						
1	1					← 1 step
1	2	1				← 2 steps
1	3	3	1			:
1	4	6	4	1		:
1	5	10	10	5	1	← five steps

-5	$1 (0.3)^5$	= .0024
-3	$5 (0.7)^1 (0.3)^4$	= .0284
-1	$10 (0.7)^2 (0.3)^3$	= .1323
+1	$10 (0.7)^3 (0.3)^2$	= .3087
+3	$5 (0.7)^4 (0.3)^1$	= .3602
+5	$1 (0.7)^5$	= .1681

Deleted  
from  
exam?

Average final position is  $5(0.7 - 0.3) = 2$   
Using formula from class

6.

Can check the formula via

$$\langle x \rangle = -5( .0024 )$$

$$- 3 ( .0284 )$$

$$- 1 ( .1323 )$$

$$+ 1 ( .3087 )$$

$$+ 3 ( .3602 )$$

$$+ 5 ( .1681 ) \Rightarrow 2.0003$$

↑ rounding error

7.

$$\boxed{4} \quad G\vec{v} = \lambda\vec{v}$$

$$G^2\vec{v} = \lambda G\vec{v} = \lambda^2\vec{v}$$

$$G^3\vec{v} = \lambda^2 G\vec{v} = \lambda^3\vec{v}$$

If  $G^3 = G$  we must have  $\lambda^3 = \lambda$

$$\lambda^3 - \lambda = 0$$

$$\lambda(\lambda^2 - 1) = 0$$

$$\lambda(\lambda+1)(\lambda-1) = 0 \quad \lambda = 0, 1, -1$$

are possible eigenvalues

Interesting factoid: These are eigenvalues

of  $S_x, S_y, S_z$  spin operators for  $S=1$ , if we set  $\hbar=1$ .

If you look at Assignment 4 it is obvious  $S_z^3 = S_z$

can also check  $S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$S_x^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad S_x^3 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = S_x !$$

[5] The  $\omega^2 = 0$  mode corresponds to a uniform

translation of all 3 masses (no springs are stretched or compressed). This works for all choices of

masses  $m_1, m_2, m_3$  and springs  $k_1, k_2$  as long as

no springs connect you to a wall, so this  $\omega^2 = 0$

$\vec{v} = (1 \ 1 \ 1)$  mode is unchanged.

Likewise the  $\omega^2 = k/m$  mode  $\vec{v} = (1 \ 0 \ -1)$

is unchanged. This corresponds to a stationary central

mass and 2 identical masses/springs to right and

left moving out of phase and hence having  $\vec{F}$  on central

mass cancel (consistent with it being stationary).

Note: I did not ask this but we expect  
the final mode to have vector  $(1 -A \ 1)$

with  $A < 2$  because a heavier central mass will  
have smaller displacement.