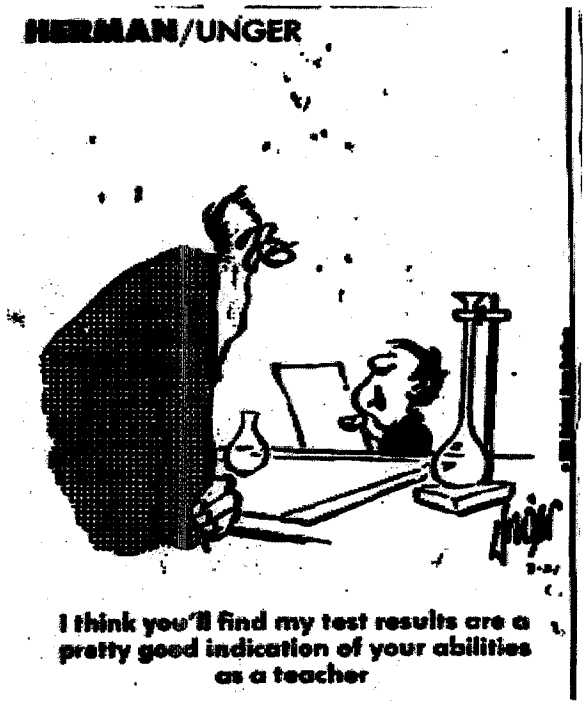


PHYSICS 104A, FALL 2015
MATHEMATICAL PHYSICS
Midterm Exam

[1.] Write down the solutions to the equation $z^6 = 2$.

[2.] In understanding the behavior of spin-1 particles in quantum mechanics, you will encounter the 3×3 matrix

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

What are the possible values you could get if you measure the x component of spin of a spin-one particle in an experiment? If your system is in the state $\vec{\psi} = (1, 0, 0)$, what are the probabilities of measuring the different possible values of S_x?

[3.] What is the definition of a projection operator?
Prove that the eigenvalues of a projection operator must be $\lambda = 0, 1$.

[4.] Use binomial theorem (left equation below) to evaluate the sum S on the right.

$$(p+q)^N = \sum_{j=0}^{N} \binom{N}{j} p^j q^{N-j} \qquad\qquad S = \sum_{j=0}^{N} \binom{N}{j} j\, p^j q^{N-j}$$

[5.] Two equal masses m are connected to each other by a spring of force constant k. One of the masses is also connected to a "wall" (ie a completely stationary object) by a spring of force constant $2k$.
(a) Write down the equations of motion $F = ma$ for the two masses.
(b) Assume a solution $x_l = v_l\, e^{i\omega t}$ for $l = 1, 2$. What equations do v_1 and v_2 obey?
(c) Compute the normal mode frequencies.
(d) Compute the normal mode vectors.
(e) In previous problems we have always found $\omega = 0$ as one of the frequencies. Did that happen here? Give a physical argument why still having $\omega = 0$ makes sense, or why you do not expect $\omega = 0$, depending on your answer.

1-1

Physics 104A
Midterm Exam Fall 2015

1 $z^6 = 2$

Write $z = re^{i\theta}$

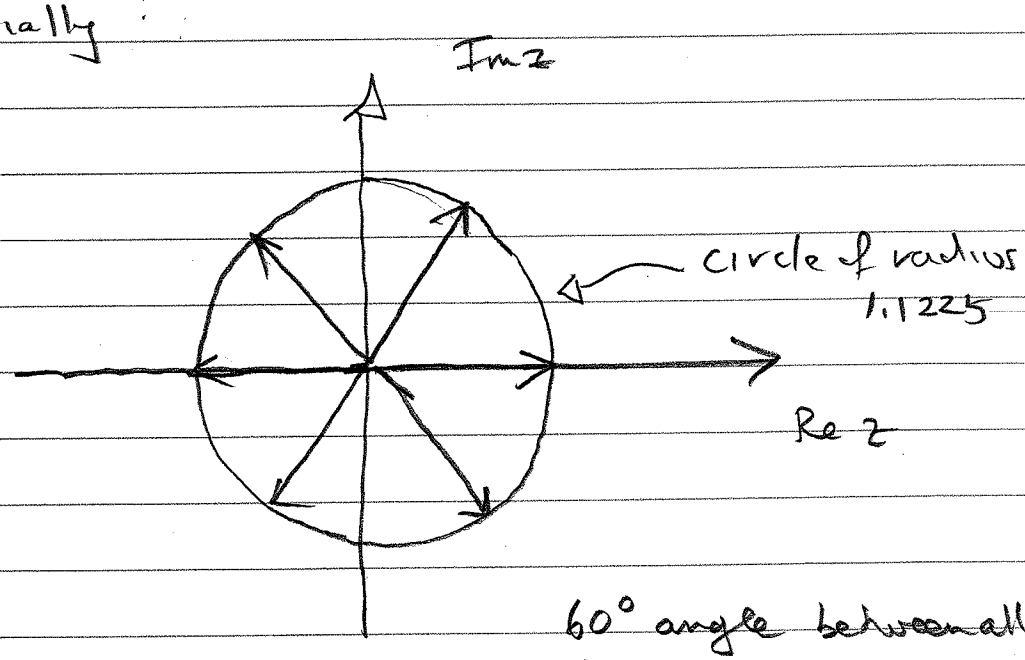
Then $r^6 e^{6i\theta} = 2$

We need $r = \sqrt[6]{2} = 1.1225$

and $6\theta = 2\pi n \qquad n = 0, 1, 2, 3, 4, 5$

so $\theta = \frac{\pi}{3}\{0, 1, 2, 3, 4, 5\}$

Pictorially:



60° angle between all arrows
(π/3 radians)

2-1

Possible values of measurement are eigenvalues of matrix representing S_x

$$0 = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 1) - 1(-\lambda)$$

$$\lambda(\lambda^2 - 2) = 0$$

$$\lambda = \{-\sqrt{2}, 0, +\sqrt{2}\} \hbar/\sqrt{2} = \{-1, 0, 1\} \hbar$$

Eigenvectors are, for $\lambda = -\sqrt{2}$

$$\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \sqrt{2}/2 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$v_1 + \sqrt{2}/2\, v_2 = 0 \qquad v_1 = -\sqrt{2}/2\, v_2$$

$$v_2 + \sqrt{2}\, v_3 = 0 \qquad v_3 = -\sqrt{2}/2\, v_2$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -\sqrt{2}/2 \\ 1 \\ -\sqrt{2}/2 \end{pmatrix} \Rightarrow \begin{pmatrix} -1/2 \\ \sqrt{2}/2 \\ -1/2 \end{pmatrix} = \vec{\phi}_1$$

2-2

for $\lambda = 0$ $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\rightarrow$ $v_1 = -v_3$ $\quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \vec{\phi}_2$

$v_2 = 0$

and for $\lambda = +\sqrt{2}$

$$\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

so that $\quad v_1 - \sqrt{2}/2\, v_2 = 0 \qquad v_1 = \sqrt{2}/2\, v_2$

$v_2 - \sqrt{2}\, v_3 = 0 \qquad v_3 = \sqrt{2}/2\, v_2$

$$1/\sqrt{2} \begin{pmatrix} \sqrt{2}/2 \\ 1 \\ \sqrt{2}/2 \end{pmatrix} \rightarrow \begin{pmatrix} 1/2 \\ \sqrt{2}/2 \\ 1/2 \end{pmatrix} = \vec{\phi}_3$$

One can verify the $\{\phi_i\}$ are $\perp$ as expected

Writing $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = a_1 \frac{1}{2} \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} + a_2 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + a_3 \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

yields

$$1 = -a_1/2 + a_2/\sqrt{2} + a_3/2$$
$$0 = a_1\sqrt{2}/2 + a_3\sqrt{2}/2$$
$$0 = -a_1/2 - a_2/\sqrt{2} + a_3/2$$

2-3

Thus $a_3 = -a_1$ and

$$0 = -a_1/2 - a_2/\sqrt{2} - a_1/2$$

So that $a_2 = -\sqrt{2}\, a_1$

and $1 = -a_1/2 - a_1 - a_1/2$

$a_1 = -1/2$

$a_2 = \sqrt{2}/2$

$a_3 = +1/2$

Prob of measuring $-\hbar \rightarrow |a_1|^2 = 1/4$

$0 \rightarrow |a_2|^2 = 1/2$

$+\hbar \rightarrow |a_3|^2 = 1/4$

3-1

A projection operator obeys $P^2 = P$

If $\quad Pv = \lambda v$

applying P to both sides

then $\quad P^2 v = \lambda P v$

using $P^2 = P$ and $Pv = \lambda v$

$$Pv = \lambda^2 v$$

using $Pv = \lambda v$

$$\lambda v = \lambda^2 v$$

Thus $\lambda^2 = \lambda$ so that $\lambda = 0$ or $\lambda = 1$

4-1

$$(p+q)^N = \sum_{j=0}^{N} \binom{N}{j} p^j q^{N-j}$$

Differentiate with respect to p

$$N(p+q)^{N-1} = \sum_{j=0}^{N} \binom{N}{j} j p^{j-1} q^{N-j}$$

Multiply by p

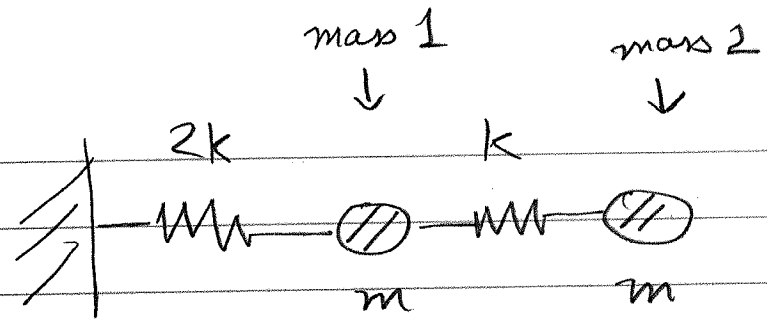
$$N(p+q)^{N-1} p = \sum_{j=0}^{N} \binom{N}{j} j p^j q^{N-j}$$

so $S = Np(p+q)^{N-1}$

Often we encounter $p+q=1$, in

which case $S = Np$

5-1



a) $$m\frac{d^2x_1}{dt^2} = -2kx_1 - k(x_1 - x_2)$$

$$m\frac{d^2x_2}{dt^2} = -k(x_2 - x_1)$$

b) $$-m\omega^2 v_1 = -2kv_1 - k(v_1 - v_2)$$

$$-m\omega^2 v_2 = -k(v_2 - v_1)$$

cancelled out $e^{i\omega t}$ in all terms

$$(3k - m\omega^2)v_1 - kv_2 = 0$$

$$-kv_1 + (k - m\omega^2)v_2 = 0$$

c) $$(3k - m\omega^2)(k - m\omega^2) - k^2 = 0$$

$$(m\omega^2)^2 - 4km\omega^2 + 3k^2 - k^2 = 0$$

$$(m\omega^2) = \left[4k \pm \sqrt{16k^2 - 4(2k^2)}\right]/2$$

$$= (2 \pm \sqrt{2})k$$

$$\omega^2 = (2 \pm \sqrt{2})k/m$$

5-2

(d) $m\omega^2 = (2+\sqrt{2})k$

$$(1-\sqrt{2})v_1 - v_2 = 0$$

$$-v_1 + (1+\sqrt{2})v_2 = 0$$ ← this eqn is just $-(1-\sqrt{2})$ times the top eqn, so they are linearly dependent as expected

$$\Rightarrow \vec{\phi}_1 = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$$

is the eigenvector.

(If you want to normalize it, divide by $\sqrt{4-2\sqrt{2}}$)

$m\omega^2 = (2-\sqrt{2})k$

$$(1+\sqrt{2})v_1 - v_2 = 0$$

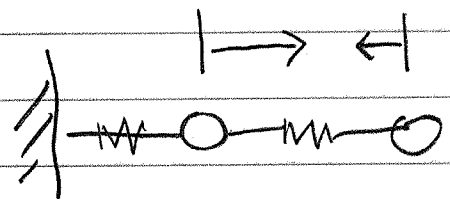
$$-v_1 - (1-\sqrt{2})v_2 = 0$$ ← again, degeneracy

$$\Rightarrow \vec{\phi}_2 = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$$ is the eigenvector

Note that $\vec{\phi}_1 \perp \vec{\phi}_2$ as expected since we have a real, symmetric matrix.

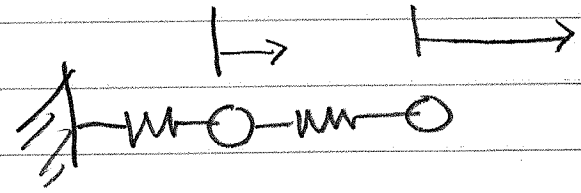
5-3

Picture $\vec{\phi}_1 = \begin{pmatrix} 1 \\ -0.414 \end{pmatrix}$ $\vec{\phi}_2 = \begin{pmatrix} 1 \\ 2.414 \end{pmatrix}$



higher ω^2

$(2+\sqrt{2})\,k/m$



lower ω^2

$(2-\sqrt{2})\,k/m$

We do not get $\omega = 0$ because the forces are not all "internal" ie part of action-reaction pairs. There is an external spring which keeps the system tethered to the wall. We cannot do a uniform translation at no energy cost.