

LA14A

What are vectors  $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  for these  $\omega^2$ ?

$$\omega^2 = 0 \quad \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

← obviously linearly dependent

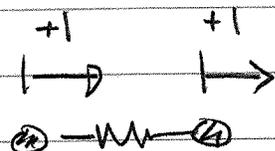
also clearly  $\det = 0$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ +1 \end{pmatrix}$$

↑

we usually

normalize



$$\omega^2 = 2k/m$$

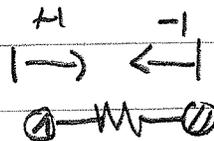
$$m\omega^2 = 2k$$

$$\begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

again obvious  $\det = 0$

and linearly dependent

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ -1 \end{pmatrix}$$



LA15

This second way of thinking generalizes better

to case of many masses. Let's do 3

$$m \ddot{x}_1 = -k(x_1 - x_2)$$

$$m \ddot{x}_2 = -k(x_2 - x_1) - k(x_2 - x_3)$$

$$m \ddot{x}_3 = -k(x_3 - x_1)$$

ADD  
EQNS

still have momentum conservation

$$m(\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3) = 0$$

$$m(\dot{v}_1 + \dot{v}_2 + \dot{v}_3) = \text{const.}$$

less clear how to get other modes,  $x_1 = v_1 e^{i\omega t}$

$$x_2 = v_2 e^{i\omega t}$$

$$x_3 = v_3 e^{i\omega t}$$

$$-m\omega^2 v_1 = -kv_1 + kv_2$$

$$-m\omega^2 v_2 = +kv_1 - 2kv_2 + kv_3$$

$$-m\omega^2 v_3 = -kv_3 + kv_1$$

LA16

$$\begin{pmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(k - m\omega^2) \left[ (2k - m\omega^2)(k - m\omega^2) - k^2 \right]$$

$$+ k \left[ -k(k - m\omega^2) - 0 \right] = 0$$

$$= (k - m\omega^2) \left[ 2k^2 - 3km\omega^2 + m^2\omega^4 - k^2 \right]$$

$$+ (k - m\omega^2) \left[ -k^2 \right]$$

$$= (k - m\omega^2) \left[ -3km\omega^2 + m^2\omega^4 \right]$$

$$= m\omega^2 (k - m\omega^2) (-3k + m\omega^2)$$

$$\omega^2 = 0$$

$$\omega^2 = k/m$$

$$\omega^2 = 3k/m$$

Q: guesses at eigenvectors?

$$\omega^2 = 0 \text{ obvious!}$$

LA17

$$\omega^2 = 0 \quad \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} k & -k & 0 \\ 0 & k & -k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$k(v_1 - v_2) = 0$$

$$v_2 = v_1$$

$$k(v_2 - v_3) = 0$$

$$v_3 = v_2$$

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

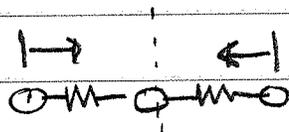
CM motion

$$m\omega^2 = k$$

$$\begin{pmatrix} 0 & -k & 0 \\ -k & k & -k \\ 0 & -k & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\hookrightarrow v_2 = 0 \quad v_1 = -v_3$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$



LA18

$m\omega^2 = 3k$  15 Homework

$$K \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$



$$\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{array} \rightarrow \begin{array}{ccc} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{array} \rightarrow \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}$$



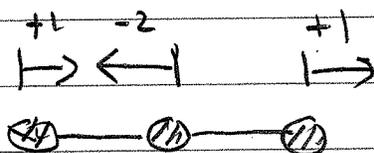
linearly dependent!

$$\rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v_2 = -2v_3$$

$$v_1 = v_3$$

$$\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$



Weird...

LA19

## General N

Q: what can you tell me ( $\omega^2 = 0$  ✓)

Not easy. We will come back to it after

discussing eigenvalues in general

Given matrix M

In words:

$$\rightarrow Mv = \lambda v$$

$v$  is eigenvector

$\lambda$  is eigenvalue

$$(M - \lambda I)v = 0$$

$$\det(M - \lambda I) = 0 \quad \text{determines eigenvalues}$$

then get eigenvectors as we saw earlier

In mass-spring problem

$$\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0 \quad \lambda = 2, 3$$

$$\lambda = 2$$

$$M - \lambda I = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

double check  $Mv = \lambda v$ !

$$\lambda = 3$$
$$M - \lambda I = \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}$$

$$\frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

LA20

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In CM the normal modes are the eigenvalues/  
eigenvectors of the "dynamical matrix".

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In QM ... all observables are associated

with Hermitian matrix. possible values

of measurement of that observable are

eigenvalues of the matrix. !

Q

Why Hermitian?

What do we expect of physical observables when  
we measure them? get real # (not complex)

(or real symmetric)

Thm Eigenvalues of Hermitian matrix are real!

$$M v = \lambda v \quad \begin{array}{l} \text{multiply by} \\ v^\dagger \text{ on left} \end{array} \rightarrow v^\dagger M v = \lambda v^\dagger v$$

$$v^\dagger M^\dagger = \lambda^* v^\dagger \rightarrow v^\dagger M^\dagger v = \lambda^* v^\dagger v$$

multiply  
by  $v$  on  
right  $\uparrow$   
but this is  $M$

So  $(\lambda - \lambda^*) v^\dagger v = 0$

$\uparrow$  either length of  $v$  is zero  
or  $\lambda - \lambda^* = 0 \Rightarrow \lambda$  is real.

LA21

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

Consider  $2 \times 2$  case + real matrices

Then Hermitian  $\rightarrow$  symmetric  $\rightarrow q = r$

$$\begin{pmatrix} p & r \\ r & s \end{pmatrix} \Rightarrow (p-\lambda)(s-\lambda) - r^2 = 0$$

$$\lambda^2 - (p+s)\lambda + ps - r^2 = 0$$

$$\lambda = \frac{1}{2} \left[ p+s \pm \sqrt{(p+s)^2 - 4(ps-r^2)} \right]$$

$$\underbrace{(p-s)^2 + 4r^2}$$

can never be negative

$\Rightarrow \lambda$  are real.

If  $q \neq r$  can easily make  $\lambda$  complex

for example  $\begin{pmatrix} p & -r \\ r & s \end{pmatrix}$

$$\lambda = \frac{1}{2} \left[ (p+s) \pm \sqrt{(p+s)^2 - 4(ps+r^2)} \right]$$

$$\underbrace{(p-s)^2 - 4r^2}$$

if  $|r| > |p-s|/2$

get negative discriminant.

LA22

of different eigenvalue

Even more interesting: Eigenvectors are  $\perp$ !

$$M v_1 = \lambda_1 v_1$$

$$\textcircled{1} v_1^+ M^+ = \lambda_1^* v_1^+ = \lambda_1 v_1^+$$

$$\textcircled{2} M v_2 = \lambda_2 v_2$$

$\uparrow$   
M

$\hookrightarrow$

because  $\lambda_1$  is real

Eqn  $\textcircled{1}$  on rhs by  $v_2$

$$v_1^+ M v_2 = \lambda_1 v_1^+ v_2$$

Eqn  $\textcircled{2}$  on lhs by  $v_1^+$

$$v_1^+ M v_2 = \lambda_2 v_1^+ v_2$$

subtract

$$(\lambda_1 - \lambda_2) (v_1^+ v_2) = 0$$

$\uparrow$

dot product of  $v_1$  and  $v_2$

Either  $\lambda_1 = \lambda_2$  or dot product = 0

$\underbrace{\hspace{10em}}$

eigenvectors  $\perp$