

LA8

$$R^T R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark \checkmark$$

general proof

$$v' = M v$$

$$(v')^T = v^T M^T \quad \leftarrow \text{remember to interchange when transposing!}$$

$$(v')^T v' = (v^T M^T)(M v)$$

$$= v^T (M^T M) v$$

matrix multiplication
is associative

$$= v^T I v$$

$$= v^T v$$

Generalization of transpose to complex matrices

is "Hermitian conjugate"

$$(A^\dagger)_{ij} \equiv A_{ji}^* \quad \text{transpose and complex conjugate}$$

If A is real $A^\dagger = A^T$

Real Matrices

Complex matrices

$$AA^T = I$$

orthogonal

$$AA^\dagger = I$$

unitary

$$A = A^T$$

symmetric

$$A = A^\dagger$$

Hermitian

Unitary matrices preserve lengths of vectors which

have complex components,

$$\sum v_n^2 = 1 \quad \text{real}$$

$$\sum |v_n|^2 = 1 \quad \text{complex}$$

$$\left. \begin{array}{l} v^T v = 1 \quad \text{real} \\ v^\dagger v = 1 \quad \text{complex} \end{array} \right\}$$

$$v' = Av$$

$$(v')^\dagger = v^\dagger A^\dagger$$

$$(v')^\dagger v' = v^\dagger \underbrace{A^\dagger A}_I v$$

LA9A

$$\det AB = \det A \det B \quad (\text{not easy to prove!})$$

$$\det M^T = \det M$$

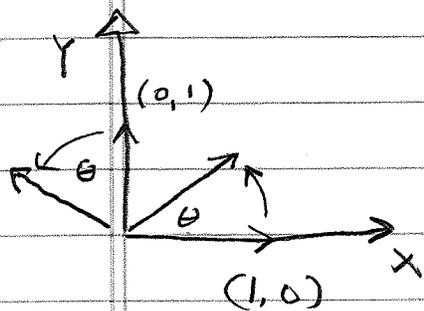
For symmetric matrix

$$\det M M^T = \det I = 1$$

$$\hookrightarrow \det M \det M^T = (\det M)^2$$

$$\therefore \det M = \pm 1$$

Rotation matrix $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ $\det M = 1$



Reflection in x-axis

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \det M = -1$$

LA 10

Matrices in QM

Vector \vec{v}

components

v_n

← discrete label
(usually finite #)

Wave function $|\psi\rangle$

components

$\psi(x)$

usual notation in QM
instead of $\vec{\psi}$

↑ continuous label
infinite #

Matrices in QM

are infinite dimensional!

Q:

$|\psi(x)|^2 =$ probability to find qm particle at x

$$\sum_n |v_n|^2 = 1 \quad \leftarrow \text{vector is normalized}$$

$$\int dx |\psi(x)|^2 = 1 \quad \leftarrow \text{sum of probabilities is 1}$$

so normalization of vector in QM
is required for interpretation of
 $|\psi(x)|^2$ as probability

$|\psi(t)\rangle$

$\psi(x,t)$ really because wave function depends on time

CM: $x(t+dt) = x(t) + v(t)dt$

$v(t+dt) = v(t) + F(x,v)/m dt$

QM

$|\psi(t+dt)\rangle \leftarrow |\psi(t)\rangle$

↑ MATRIX gives vector

$|\psi(t+dt)\rangle$ from vector $|\psi(t)\rangle$

LA10A

Q Nature of Matrix

If $|\psi(t)\rangle$ has length 1

we want $|\psi(t+dt)\rangle$ to have length 1 also

\Rightarrow Matrix must be unitary.

$$|\psi(t+dt)\rangle = U|\psi(t)\rangle$$

We will follow up more on this later. but for now,

connect to your Homework

$$e^{-it\sigma_x}$$

$$e^{-it\sigma_y}$$

$$e^{-it\sigma_z}$$



It turns out these matrices
are some of the ones that evolve
spin $\frac{1}{2}$ wave functions.

You are computing these.

They are all unitary

you can verify this

Spin $\frac{1}{2}$ is a very nice QM problem because

unlike $\psi(x)$ infinite dim - spin $\frac{1}{2}$ is 2 dim.

LA11

We saw (when we were young) matrices,
arising in linear algebra problems. They also
arise in QM (as we just started to see)
and in CM. Let's look at CM first.

We did one mass on a spring. Now 2:

$$\left. \begin{aligned} m_1 \ddot{x}_1 &= -k(x_1 - x_2) \\ m_2 \ddot{x}_2 &= -k(x_2 - x_1) \end{aligned} \right\} \begin{array}{l} \text{Newton's} \\ \text{third laws.} \end{array}$$

← "internal force"

One (non matrix) approach

$$\text{Add } m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = 0$$

$$m_1 \dot{x}_1 + m_2 \dot{x}_2 = C \quad \leftarrow \text{what is this?}$$

$$\frac{d}{dt} (m_1 \dot{x}_1 + m_2 \dot{x}_2) = \frac{d}{dt} (m_1 v_1 + m_2 v_2) = 0$$

Momentum conservation. ← no "external forces"

LA12

Set $m_1 = m_2 = m$ for simplicity

Subtract $m \ddot{x}_1 = -k(x_1 - x_2)$
 $m \ddot{x}_2 = -k(x_2 - x_1)$

$$m(\ddot{x}_1 - \ddot{x}_2) = -2k(x_1 - x_2)$$

$$\frac{d^2}{dt^2}(x_1 - x_2) = -\frac{2k}{m}(x_1 - x_2)$$

"relative coordinate" $x = x_1 - x_2$

$$\ddot{x} = -\frac{2k}{m}x$$

Oscillator with $\omega^2 = 2k/m$

"center of mass coordinate" $\bar{x} = \frac{x_1 + x_2}{2}$

$$\ddot{\bar{x}} = 0 \quad \bar{x} = C + Dt$$

$$x(t) = x_1(t) - x_2(t) = A \cos \omega t + B \sin \omega t$$

$$\bar{x}(t) = x_1(t) + x_2(t) = C + Dt$$

what determines A, B, C, D ? $x_1(0)$

$x_2(0)$

The initial conditions

$\dot{x}_1(0)$

$\dot{x}_2(0)$

$m \ddot{x} = -\omega^2 x$ has oscillating solns $\cos \omega t, \sin \omega t$

except when $\omega = 0$ $\ddot{x} = 0 \Rightarrow x = C + Dt$

LA 13

$$x_1(0) - x_2(0) = A$$

$$x_1(0) + x_2(0) = C$$

$$\dot{x}_1(0) - \dot{x}_2(0) = \omega B$$

$$\dot{x}_1(0) + \dot{x}_2(0) = D$$

Then, ~~of~~ course

$$x_1(t) = \frac{1}{2} (x(t) + \bar{x}(t))$$

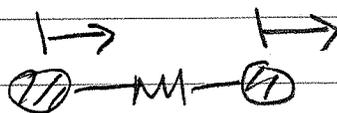
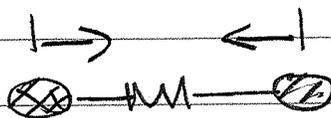
$$x_2(t) = \frac{1}{2} (-x(t) + \bar{x}(t))$$

How to generalize to many masses?

This is best done with matrices.

preview of:

"Normal modes"



LA14

Let's redo 2 mass case

Guess solution $x_1 = v_1 e^{i\omega t}$ \leftarrow same ω !
 $x_2 = v_2 e^{i\omega t}$ \leftarrow will it work?
 Its just a guess of ours (not obvious)

$$-m\omega^2 v_1 e^{i\omega t} = -k(v_1 - v_2) e^{i\omega t}$$

$$-m\omega^2 v_2 e^{i\omega t} = -k(v_1 - v_2) e^{i\omega t}$$

$e^{i\omega t}$
cancel!

$$\begin{pmatrix} k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Q

det Must vanish

$$m(k - m\omega^2)^2 - k^2 = 0$$

$$k - m\omega^2 = \pm k$$

$$m\omega^2 = 0, 2k$$

$$\omega^2 = 0$$

$$\omega^2 = 2k/m$$

Q

Does this remind any one of anything?

EIGENVALUES