

PHYSICS 104A, FALL 2015
MATHEMATICAL PHYSICS
Final Exam

[1.] A quantum mechanical particle has initial wave function $\psi(x, 0) = ne^{-a|x|/2}$. What value of n is needed for normalization? What is the distribution of momentum $c(k)$? Check that $c(k)$ is normalized. Compute either Δx or Δp .

[2.] A string is stretched from $x = 0$ to $x = L$. Its displacement from equilibrium at position x and time t is denoted by $y(x, t)$ and obeys the wave equation,

$$\frac{\partial^2 y(x, t)}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y(x, t)}{\partial t^2} = 0$$

(a) Construct the general solution by the separation of variables method. How do you use the fact that the string is tied down at $x = 0$ and $x = L$ to narrow down the possible solutions? (b) Compute $y(x, t)$ if you are also told the initial displacement and velocity,

$$y(x, t=0) = \delta(x - L/2) \quad \left. \frac{\partial y}{\partial t}(x, t) \right|_{t=0} = 0$$

[3.] (i) On what sort of object does the gradient operator act? What information does it tell you? (ii) On what sort of object does the divergence operator act? What information does it tell you? (iii) On what sort of object does the curl operator act? What information does it tell you? Feel free to draw some pictures in your response to these questions.

[4.] Compute $\int \vec{F} \cdot d\vec{l}$ of the vector field $\vec{F} = -x\hat{i} + y^2\hat{j}$ from the origin $(0, 0)$ to $(2, 4)$ along (i) a straight line path; and (ii) along a parabolic path $y = x^2$. Comment on the relationship between your two answers.

[5.] Compute the eigenvalues, normalized eigenvectors, and inverse of the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

[6.] Consider vectors \vec{v} in the 2D (xy) plane. The operator \mathcal{O} triples the length of \vec{v} and also rotates it counterclockwise by 30° to produce the vector \vec{w} . What is the matrix for \mathcal{O} ? That is, find a matrix M so that $\vec{w} = M\vec{v}$.

Physics 104A Fall 2015
Final Exam Solution

$$[1] \quad \psi(x, 0) = \int dk c(k) e^{ikx} / \sqrt{2\pi}$$

$$c(k) = \int dx \psi(x, 0) e^{-ikx} / \sqrt{2\pi}$$

To normalize $\psi(x, 0)$:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\psi(x)|^2 dx = n^2 \int_{-\infty}^{\infty} e^{-ax^2} dx \\ &= 2n^2 \int_0^{\infty} e^{-ax^2} dx = 2n^2/a \end{aligned}$$

$$\therefore n = \sqrt{a/2}$$

$$\begin{aligned} \text{Then } c(k) &= \sqrt{\frac{a}{4\pi}} \int_{-\infty}^{\infty} dx e^{-ax^2/2} e^{-ikx} \\ &= \sqrt{\frac{a}{4\pi}} \left\{ \int_{-\infty}^0 dx e^{ax^2/2} e^{-ikx} + \int_0^{\infty} e^{-ax^2/2} e^{-ikx} dx \right\} \\ &= \sqrt{\frac{a}{4\pi}} \left\{ \frac{e^{x(a/2 - ik)}}{(a/2 - ik)} \Big|_0^\infty + \frac{e^{x(-a/2 - ik)}}{(-a/2 - ik)} \Big|_0^\infty \right\} \\ &= \sqrt{\frac{a}{4\pi}} \left\{ \frac{1}{a/2 - ik} + \frac{1}{a/2 + ik} \right\} \\ &= \sqrt{\frac{a}{4\pi}} \left\{ \frac{a/2 + ik + a/2 - ik}{a^2/4 + k^2} \right\} \\ &= \sqrt{\frac{a}{4\pi}} \frac{a}{(a^2/4 + k^2)} \end{aligned}$$

Normalization of $c(k)$

$$N = \int_{-\infty}^{\infty} |c(k)|^2 dk = \frac{a}{4\pi} \int_{-\infty}^{\infty} dk \frac{a^2}{(\frac{a^2}{4} + k^2)^2}$$

$$\text{Let } k = a/2 \tan \theta \quad dk = \frac{a}{2} \sec^2 \theta d\theta$$

$$\frac{a^2}{4} + k^2 = \frac{a^2}{4} (1 + \tan^2 \theta) = \frac{a^2}{4} \sec^2 \theta$$

$$N = \frac{a}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{a^2 \cdot \frac{a}{2} \sec^2 \theta d\theta}{\frac{a^4}{16} \sec^4 \theta}$$

$$= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta = \frac{2}{\pi} \pi \frac{1}{2} = 1. \quad \blacksquare$$

$$\langle x \rangle = \langle p \rangle = 0 \quad \text{by symmetry, eg}$$

$$\langle p \rangle = \hbar \langle k \rangle = \int_{-\infty}^{\infty} dk k |f(k)|^2 = \phi.$$

\uparrow even function
 \downarrow odd function

$$\Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx$$

$$= \int_{-\infty}^{\infty} x^2 n^2 e^{-ax} dx$$

$$= 2 \int_0^{\infty} n^2 x^2 e^{-ax} dx$$

$$\begin{aligned}
 \langle x^2 \rangle &= 2 \frac{a}{2} \int_0^\infty x^2 e^{-ax} dx \\
 &= a \left\{ x^2 e^{-ax} \Big|_{-a}^{\infty} + \frac{1}{a} \int_0^\infty 2x e^{-ax} dx \right\} \\
 &= 2 \times \frac{e^{-ax}}{-a} \Big|_0^\infty + \frac{2}{a} \int_0^\infty e^{-ax} dx
 \end{aligned}$$

$$\langle x^2 \rangle = 2/a^2$$

$$\Delta x = \sqrt{2}/a.$$

2-1

$$\text{Wave egn} \quad \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

$$\text{where } x = \sqrt{\frac{T}{\mu}}$$

solve by separation of variables

$$y(x, t) = f(x)g(t)$$

$$f''(x)g(t) = \frac{1}{v^2} f(x)g''(t)$$

$$\frac{f''(x)}{f(x)} = \frac{1}{v^2} \frac{g''(t)}{g(t)} = -k^2$$

$$f(x) = \begin{cases} \sin kx & g(t) = \begin{cases} \sin wt & \text{where} \\ \cos kx & w = kv \end{cases} \end{cases}$$

Since $y(x=0, t) = 0$ for all t , the

solution $f(x) = \cos kx$ is not allowed.

Similarly $y(x=L, t) = 0$ for all t restricts

$$k = \frac{n\pi}{L}$$

$$y(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left\{ a_n \cos \frac{n\pi v t}{L} + b_n \sin \frac{n\pi v t}{L} \right\}$$

This follows because wave egn is linear, so can form arbitrary linear combinations of solutions for different k .

The initial condition

$$0 = \frac{dy}{dt}(x, t=0) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \frac{n\pi v}{L} \left\{ -a_n \sin \frac{n\pi t}{L} + b_n \cos \frac{n\pi t}{L} \right\}$$

forces $b_n = 0 \quad \forall n$. Setting $t=0$ we then have

$$y(x, 0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

To determine a_n , multiply both sides by $\sin \frac{m\pi x}{L}$

and integrate $\int_0^L dx$. Use $\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \frac{L}{2} \delta_{nm}$

$$\int_0^L \sin \frac{m\pi x}{L} y(x, 0) dx = a_m L/2$$

\uparrow
 $\delta(x - L/2)$

$$\text{So we see } a_m = \frac{2}{L} \sin \frac{m\pi}{2}$$

$$a_1 = 2/L$$

$$a_2 = 0$$

$$a_3 = -2/L$$

$$a_4 = 0 \quad \text{etc}$$

a_m does not

decay at all

with m because

a perfectly localized
pulse in real space
requires a completely
extended representation



in Fourier space

Answer to 7a.

(deleted from exam)

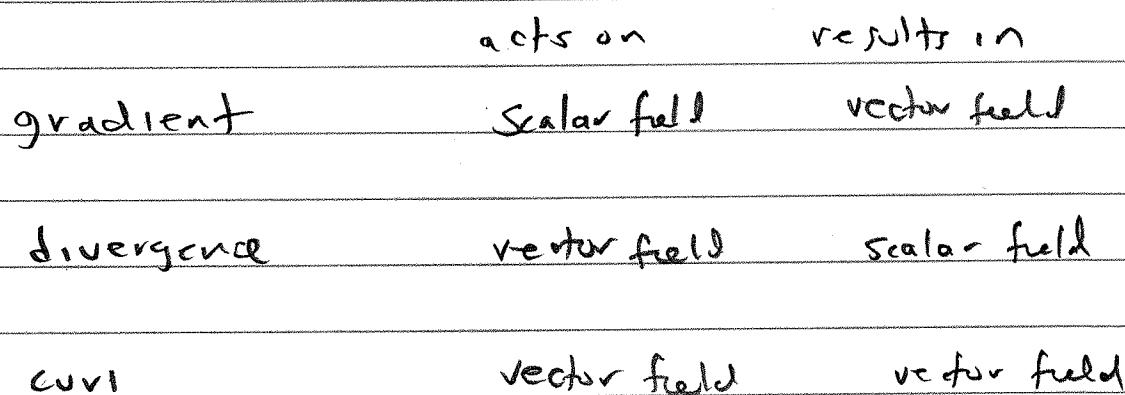
2-3

$$y(x,t) = \frac{z}{L} \sum_{n \text{ odd}}^{N-1} (-1)^{\frac{n-1}{2}} \sin \frac{n\pi x}{L} \cos \frac{n\pi vt}{L}$$

3-1

Scalar field : A quantity assigning a real number (scalar) to all points in space

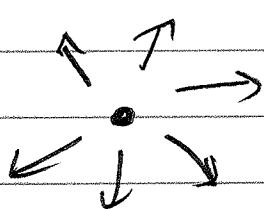
Vector field : A quantity assigning a vector to all points in space



gradient tells you spatial direction in which scalar field is increasing most rapidly

divergence tells you if vectors are "spreading apart" spatially (see picture)

curl tells you if vectors are "rotating around" a given region

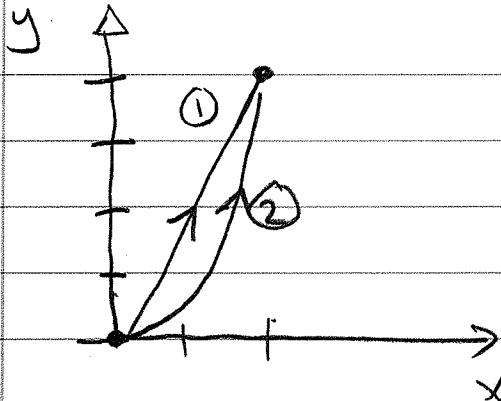


Divergence will
be big; curl will
be small



curl will be big;
divergence will
be small

4-1



$$\vec{F} = -x\hat{i} + y^2\hat{j}$$

$$\textcircled{1} \quad x(t) = t \quad 0 < t < 2$$

$$y(t) = 2t$$

$$\textcircled{2} \quad x(t) = t \quad 0 < t < 2$$

$$y(t) = t^2$$

$$\textcircled{1} \quad \int \vec{F} \cdot d\vec{r} = \int_{\text{curve}} F_x dx + F_y dy$$

$$= \int -x dx + y^2 dy$$

$$= \int_0^2 dt \left[-t dt + 4t^2 2dt \right]$$

$$= -t^2/2 + 8t^3/3 \Big|_0^2 = -2 + 64/3 = 58/3$$

$$\textcircled{2} \quad = \int_0^2 dt \left[-t dt + t^4 2t dt \right]$$

$$= -t^2/2 + 2t^6/6 \Big|_0^2 = -2 + 128/6 = 58/3$$

5-1

Eigenvalues

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)[\lambda^2 - 1] = 0$$

$$(3-\lambda)(\lambda+1)(\lambda-1) = 0$$

$$\lambda = 3, \lambda = -1, \lambda = +1$$

Eigenvectors $\lambda = 3$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} y=0 \\ x=0 \end{matrix} \Rightarrow v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda = 1$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} -x+y=0 \\ x-y=0 \\ 2z=0 \end{matrix} \quad v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = -1 \text{ similarly } \vec{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Inverse: Take advantage of fact that M is block diagonal (could do this in earlier parts of problem as well!)

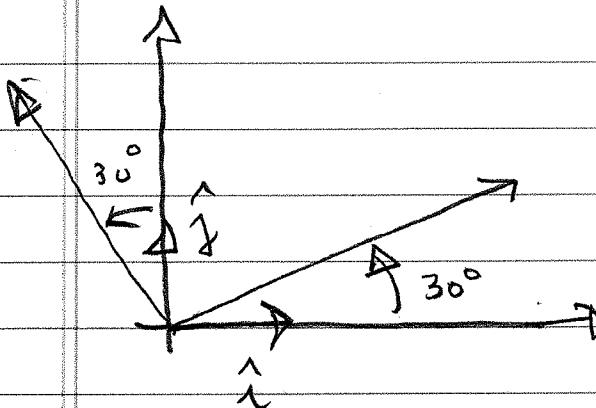
$$\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right) \quad M^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

$\hat{\in} M^{-1}$

6-1

When \hat{O} acts on $\hat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ it first converts it to $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

and then to $\begin{pmatrix} 3 \cos 30^\circ \\ 3 \sin 30^\circ \end{pmatrix} = \begin{pmatrix} 3\sqrt{3}/2 \\ 3/2 \end{pmatrix}$



When \hat{O} acts on $\hat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ one gets $\begin{pmatrix} -3/2 \\ 3\sqrt{3}/2 \end{pmatrix}$

so $M = \begin{pmatrix} 3\sqrt{3}/2 & -3/2 \\ 3/2 & 3\sqrt{3}/2 \end{pmatrix}$.

7b-1

7a is answered on page 2-2 ↴ (deleted from exam)

$$M = \begin{pmatrix} 3\sqrt{2}/2 & -3/2 \\ 3/2 & 3\sqrt{2}/2 \end{pmatrix}$$

Eigenvalues $(3\sqrt{2}/2 - \lambda)^2 + (3/2)^2 = 1$

are complex! This is okay because M is not Hermitian (real symmetric).

M has no real eigenvectors. This is

physically reasonable. If the operator $\hat{\vec{O}}$ rotates vectors there is no way $\overset{\wedge}{\vec{O}} \vec{v}$ can be parallel to \vec{v} .