Final Exam, Physics 104A, Fall 2016

Please do **only five** of the following **six** problems.

[1.](a) What is the definition of a Hermitian matrix? Give an example of dimension 2 that contains no zero entries. (b) What is true of the eigenvalues and eigenvectors of a Hermitian matrix? (c) What is the definition of a orthogonal matrix? (d) Suppose $\mathcal{O}$ is an orthogonal matrix and $\vec{w} = \mathcal{O}\,\vec{v}$ where $\vec{v}$ and $\vec{w}$ have *real* components. What property does $\vec{w}$ inherit from $\vec{v}$? (e) Give a physical context in which orthogonal matrices arise, and briefly explain your answer. (f) Extra Credit: Prove your answer to (d).

[2.] Compute the Fourier Series for a periodic function $f(x+8) = f(x)$, with $f(x) = 6$ when $0 < x < 2$ and $f(x) = 0$ when $2 < x < 8$. Sketch $f(x)$ and the contributions of the a_0, a_1, and b_1 terms. (Note: when you add these up they should look fairly reasonable, but not quite as close to $f(x)$ as some of the other examples we have done.)

[3.] (a) Give an example of two matrices A and B which do not commute when multiplied. (b) Prove that $(AB)^T = B^T A^T$. Carefully justify *each* step of your proof.

[4.] (a) Write the matrix for the operator which projects a vector onto the line $y = x/3$. Use as a basis $\hat{e}_1$, the unit vector parallel to the $\hat{x}$ axis, and $\hat{e}_2$, the unit vector parallel to the $\hat{y}$ axis. (b) Suppose instead you change basis to $\hat{f}_1 = (3\,\hat{e}_1 + \hat{e}_2)\,/\,\sqrt{10}$ and $\hat{f}_2 = (-\hat{e}_1 + 3\,\hat{e}_2)\,/\,\sqrt{10}$. What is the matrix for the operator now? Interpret your answer.

[5.] A string is stretched from $x = 0$ to $x = L$. Its displacement from equilibrium at position x and time t is denoted by $y(x,t)$, and obeys the wave equation,

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

(a) Construct the general solution by the separation of variables method. How do you use the fact that the string is tied down at $x = 0$ and $x = L$ to narrow down the possible solutions? (b) Compute $y(x,t)$ if you are also told the initial displacement and velocity,

$$y(x, t=0) = 0 \qquad\qquad \frac{\partial y}{\partial t}(x,t)\Big|_{t=0} = 4k\,x\,(L-x)$$

<u>Useful integrals:</u> $\int x^2 \sin kx\, dx = -x^2 \cos kx/k + 2\,x \sin kx/k^2 + 2 \cos kx/k^3$.
$\int x \sin kx\, dx = -x \cos kx/k + \sin kx/k^2$.

[6.] Solve Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

inside the rectangular box $0 < x < L$ and $0 < y < H$. The edges of the box have the potentials shown in the figure.