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Intro BS UCI 1982
PhD UCSB 1986
UCD 1989
Computational CMP
Morning basketball 7:15 AM
MWF ARC

Mathematical Physics

Course website has organizational details

You should have received email with link

Intimate connection between Mathematics and Physics

[Q] Classic example is Newton : Calculus + Gravity

Today many continuing examples : investigations of superconducting & magnetic phase transitions, to topological insulators etc rely on "tensor product states" to describe many e^- wave function.

My goal here is to establish both "classic" math physics techniques as well as more recent ones, and to connect to research.

Begin with elementary topics

complex #'s

sequences, series

linear algebra

vector spaces

Fourier Series

Legendre Series

→ partial differential Eqs. (Maxwell, Poisson, Schrödinger, ...) Laplace

Some numerical work (connect to 102)

C1//

Part of history of mathematics is enlargement of quantities used to describe/solve eqns. Matrices/non-commuting objects required for QM!

Complex Numbers	Sticking to numbers
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Positive integers don't allow solns to $n+7=2$
so introduce negative integers

If all you know about are whole #'s then

$$3x = 2$$

has no soln. You introduce "rational #'s" like $x = \frac{2}{3}$

to provide solns. Whole #'s \in rational #'s. Free

to introduce any (consistent) set of rules to manipulate

rational #'s that you like, eg demand commutative

law of addition, multiplication etc.

↓
skip

$$3x = 2 \quad x = \frac{2}{3}$$

$$7y = 4 \quad y = \frac{4}{7}$$

↓ ↓

$$(3x)(7y) = 2(4)$$

$$21xy = 8$$

$$xy = \frac{8}{21} = \frac{2}{3} \frac{4}{7} \Rightarrow \text{when you multiply}$$

rational #'s you multiply numerators

and denominators.



otherwise soln of $3x=2$ $7y=4$

would not obey usual rules for multiplying eqns.

CIA/1

Same for complex #'s

$$z^2 = 4$$

Q

Did we skip any steps
in going from rational #'s
to complex?
Yes! 2 steps: irrational #'s
 $x^2 = 2$
and transcendental #'s.

has no soln. So introduce new # $i^2 = -1$

Then $z = 2i$ will solve $z^2 = 4$.

$$z^2 + 4z + 13 = 0$$



quadratic eqn

$$z = \frac{-4 \pm \sqrt{16 - 52}}{2}$$

comes from completing
the square ie

$$= -2 \pm 3i$$

Imaginary part

$$w = a+bi$$

$$z = c+di$$

"Standard" mathematical
manipulations

$$w+z = (a+c) + i(b+d)$$

$$w \cdot z = (ac - bd) + i(ad + bc)$$

Q

$$\frac{w}{z} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd) + i(bc-ad)}{c^2 + d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + i \frac{bc-ad}{c^2+d^2}$$

Sum, product, quot.
of complex numbers
are complex numbers.

C2//

Further illustration of manipulating complex #'s

De Moivre's Theorem

Recall Taylor expansions

$$e^x = 1 + x + x^2/2 + x^3/6 + x^4/24 + \dots$$

$$\left(\text{check: } \frac{d}{dx} e^x = 1 + x + x^2/2 + x^3/6 + \dots \right)$$

$$\sin x = x - x^3/6 + x^5/120 - \dots$$

$$\cos x = 1 - x^2/2 + x^4/24 - \dots$$

$$\left(\text{check } \frac{d}{dx} \sin x = \cos x ! \right)$$

$$e^{ix} = \cos x + i \sin x$$

$$e^{i\theta} = 1 + i\theta + i^2/2(-\theta)^2 + i^3/6(-\theta)^3 + i^4/24(-\theta)^4 + \dots$$

$$= (1 - \theta^2/2 + \theta^4/24 + \dots) + i(\theta - \theta^3/6 + \dots)$$

$$= \cos \theta + i \sin \theta$$

Special cases

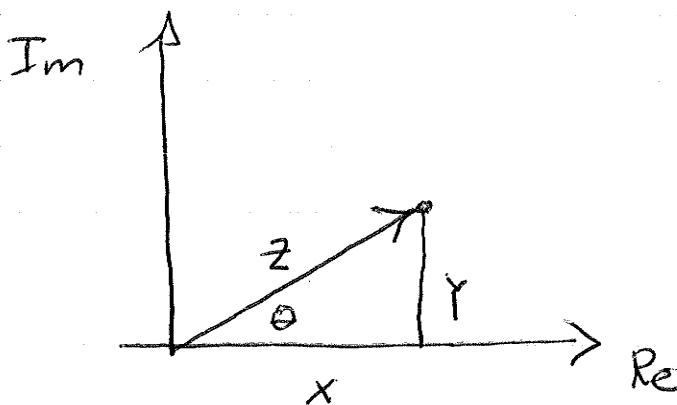
$$e^{i\pi} = -1$$

$$e^{i\pi/2} = i$$

$$e^{2\pi i} = 1 \quad \text{etc}$$

Q3//

Graphical Representation of Complex #'



$$z = x + iy$$

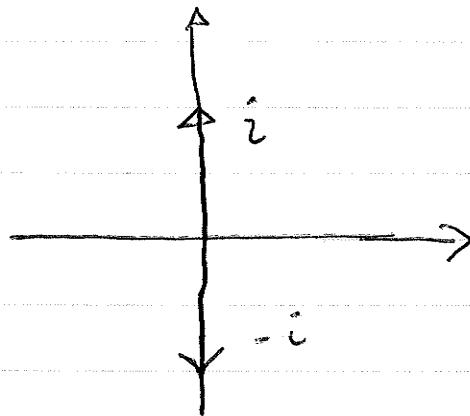
Natural to use polar coordinates

$$r = |z| = \sqrt{x^2 + y^2}$$

$$z = re^{i\theta}$$

$$\begin{aligned} &= r(\cos\theta + i\sin\theta) \\ &= x + iy \end{aligned}$$

In particular



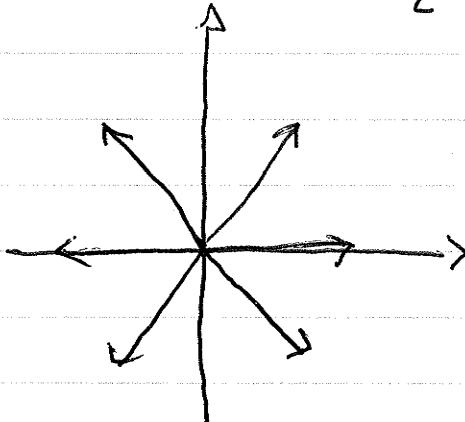
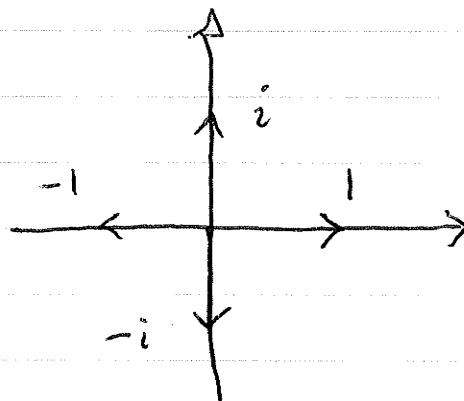
Two vectors $\pm i$

are solns of $z^2 = 1$

Sols of $z^4 = 1$ $1, i, -1, -i$

$$\boxed{Q3:}$$

$$z^6 = 1$$



C2B //

[Q] Derivation? Use polar representation

$$z^4 = 1 = e^0 e^{2\pi i} e^{4\pi i} e^{6\pi i}$$

$$z = e^0 e^{i\pi/2} e^{i\pi} e^{i3\pi/2}$$

$$1 \quad i \quad -1 \quad -i$$

[Q] Why not $e^{8\pi i}, e^{10\pi i}, \dots$?

Similarly $z^6 = 1$

$$z = e^0 e^{i\pi/3} e^{i2\pi/3} e^{i\pi} e^{i4\pi/3} e^{i5\pi/3}, \dots$$

In general $z^N = 1$ has solns

$$z = e^{i2\pi l/N} \quad l = 0, 1, 2, \dots, N-1 \quad \left. \begin{array}{l} \text{or } l = 1, 2, \dots, N \\ \text{equivalent} \end{array} \right\}$$

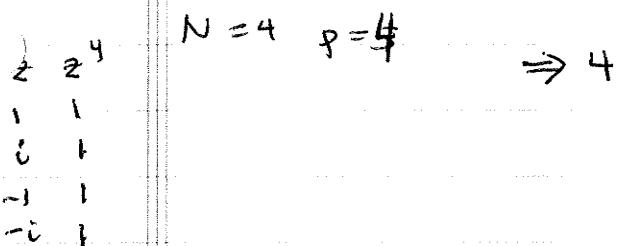
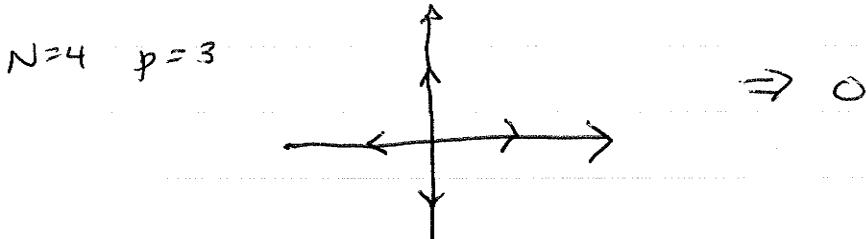
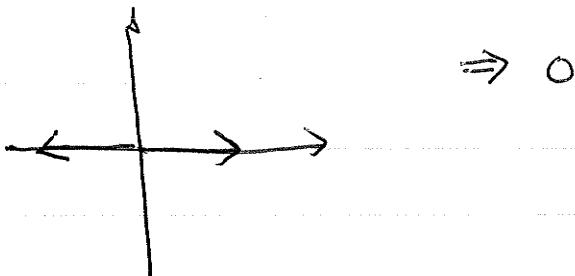
Notice that $\sum \{\text{all roots of } z^N = 1\} = \emptyset$

Usual rules for "vector addition"

c2c //

integer

What if you raise roots of $z^N=1$ to some power p and sum them? we saw if $p=1$ we get ϕ .



In general $\sum (\text{roots of } z^N=1)^p = \phi \left\{ \begin{array}{l} \text{unless } p = \text{multiple} \\ \text{of } N \\ \sum_{k=1}^N \left(e^{i \frac{2\pi k}{N}} \right)^p = \phi \end{array} \right\}$

Who cares?

(1) We will use this when we diagonalize matrices!

\vec{k}
3D generalization of wavevector \vec{k}

(2) When rays are incident on solid only certain $\vec{k} + \vec{p}$ will come out.



The allowed \vec{p} are 3D generalization of condition $p = 0, N, 2N, \dots$ BRAGG scattering determines crystal structure.

C2D//

For periodic crystal atomic positions

$$\vec{R} = n_1 \vec{q}_1 + n_2 \vec{q}_2 + n_3 \vec{q}_3 \quad n_1, n_2, n_3 \text{ integers}$$

For what \vec{p} are

$$\sum_{n_1, n_2, n_3} e^{i\vec{p} \cdot \vec{R}} \neq 0 ?$$

C3 //

Proof of wisdom of introducing new mathematical

objects is whether they are useful. Here's another example:

$$e^{2i\theta} = \cos 2\theta + i \sin 2\theta$$

$$e^{2i\theta} = e^{i\theta} e^{i\theta} = (\cos \theta + i \sin \theta)^2$$

$$= (\cos^2 \theta - \sin^2 \theta) + i(2 \sin \theta \cos \theta)$$

i. $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

We motivated complex #'s as allow us to solve general quadratic eqn.

2 complex #'s are equal iff their real / imag parts are equal.

By far the easiest proof of double angle formulae in trig!

Skip

↓ Connection between $e^{i\theta}$ and $\sin \theta / \cos \theta$

established by de Moivre theorem. Consider also

$$\text{"hyperbolic sin"} \rightarrow \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\sinh(i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{\cos \theta + i \sin \theta - (\cos \theta - i \sin \theta)}{2}$$

$$\sinh i\theta = i \sin \theta$$

so hyperbolic sin/cos closely related to "ordinary" sin/cos

C4//

One of best examples of utility of complex #'s
in physics is in soln of damped harmonic oscillator

$$\begin{aligned} F &= ma \\ \rightarrow & \\ -kx - \gamma v & \\ \text{fraction} & \end{aligned}$$

$$m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0$$

$$x = A e^{i\omega t}$$

$$\frac{dx}{dt} = A i\omega e^{i\omega t}$$

$$\frac{d^2x}{dt^2} = -\omega^2 A e^{i\omega t}$$

$$A e^{i\omega t} (-m\omega^2 + i\gamma\omega + k) = 0$$

$$\omega = \frac{-i\gamma \pm \sqrt{-\gamma^2 + 4km}}{-2m}$$

$$\omega = \frac{i\gamma}{2m} \pm \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$$

C5//

$$x = A e^{i \omega t} = A e^{-\gamma t / 2m} e^{\pm i \sqrt{k/m - \gamma^2/4m^2} t}$$

Q:

Underdamped
critically damped
overdamped

damping
(decaying
exponential)

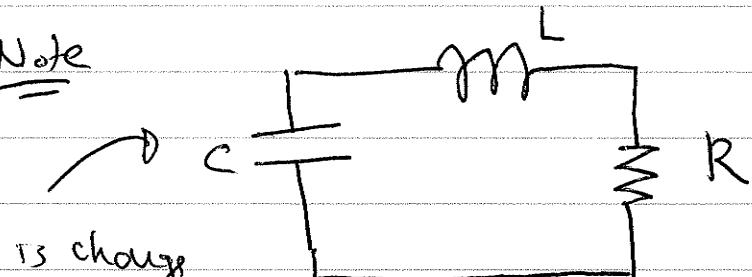
usual sine/cosine
except $\sqrt{k/m} = \omega_0$
frequency is
reduced to

does anyone recall
meaning?

$$\sqrt{k/m - \gamma^2/4m^2}$$

Solution w/o using complex #'s is much more tedious.

Note



Q is charge
on capacitor
plate

$$+L \frac{dI}{dt} + IR + \frac{Q}{C} = 0$$

$$I = \frac{dQ}{dt}$$

$$L \leftrightarrow m$$

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = 0$$

$$R \leftrightarrow \gamma$$

$$m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0$$

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

$$-Rt/2L \pm i\sqrt{1/LC - R^2/4L^2} t$$

$$Q = A e^{-Rt/2L} e^{i\sqrt{1/LC - R^2/4L^2} t}$$

check units! $L \frac{d^2Q}{dt^2} : IR \quad \frac{[L]}{T^2} [Q] : \frac{Q}{[R]} \quad \frac{[R]}{[L][C]} = T^{-1}$

$$L \frac{d^2Q}{dt^2} : Q/C$$

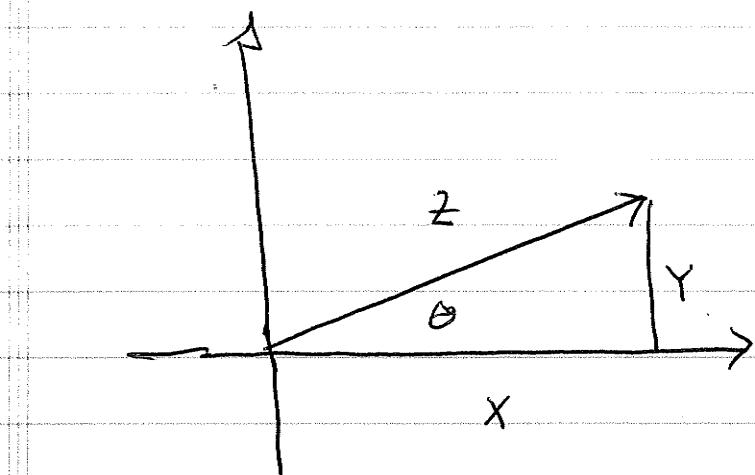
$$\frac{[L]}{T^2} [Q] : Q/[C]$$

$$\frac{1}{[L][C]} = T^{-2}$$

C6//

~~Plane anterior CIB~~

Graphical representation of complex #'s



$$z = x + iy$$

$$= r \cos \theta + i \sin \theta$$

$$|z|$$

$$= r e^{i\theta}$$

C7A

Driven mass-spring system

$$F_0 \cos \omega t = m \frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + kx$$

$$\stackrel{?}{=} \operatorname{Re} F_0 e^{i\omega t}$$

d/dt doesn't mix Re and Im pieces of function

Assume $x = x_0 e^{i\omega t}$ and take Real part at end

NB this is not ω_0 the natural

Q why?
with damping
oscillations at
natural frequency
die out "transients"

$$F_0 e^{i\omega t} = x_0 e^{i\omega t} \{ -\omega^2 m + i\gamma\omega + k \}$$

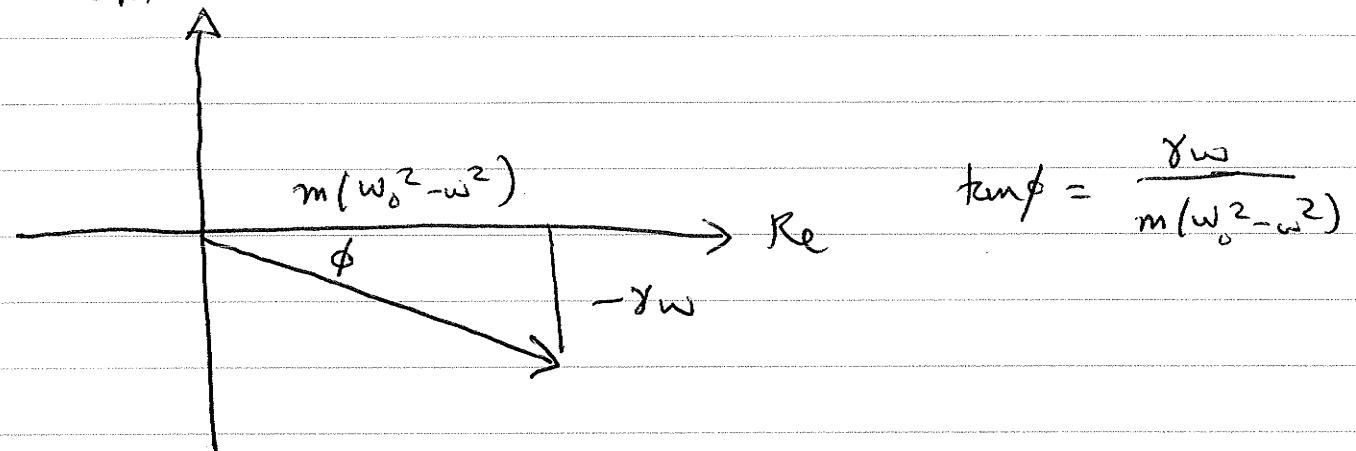
$$x_0 = \frac{F_0}{k - \omega^2 m + i\gamma\omega} = \frac{F_0}{m(\omega_0^2 - \omega^2) + i\gamma\omega}$$

Define

$$\omega_0^2 \equiv k/m = \frac{F_0}{m(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \quad m(\omega_0^2 - \omega^2) - i\gamma\omega$$

C7B

Im



$$m(w_0^2 - \omega^2) - i\gamma\omega = \sqrt{m^2(w_0^2 - \omega^2)^2 + \gamma^2\omega^2} e^{-i\phi}$$

$$x_0 = \frac{F_0}{\sqrt{m^2(w_0^2 - \omega^2)^2 + \gamma^2\omega^2}} e^{-i\phi}$$

$$x(t) = \frac{F_0}{\sqrt{m^2(w_0^2 - \omega^2)^2 + \gamma^2\omega^2}} e^{i(\omega t - \phi)}$$

"Resonance" if $\omega \rightarrow w_0$

Response is big

[Q]

$$x(t) \sim \cos(\omega t - \phi)$$

Does someone remember/know how to incorporate initial conditions

$$x(t=0)$$

$$v(t=0)$$

into sol'n?

Mars lags the force by phase ϕ

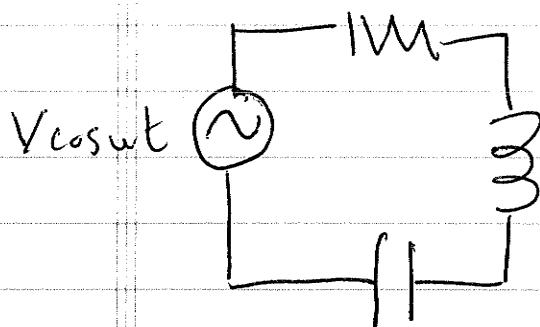
Very important in Electrical circuits especially (see NW)

[A]

transient piece
(sol'n to eqn with $F_0 = 0$)

C7/1

Driven LRC circuit



$$V_{\text{coswt}} = L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q$$

↑
Re $V e^{i\omega t}$

$\frac{d}{dt}$ doesn't mix Re + Im pieces of function

i. assume $Q = A e^{i\omega t}$ and take Re part at end

↑

Note this is driving frequency

not $\omega_0 = 1/\sqrt{LC}$ natural frequency

$$V e^{i\omega t} = A e^{i\omega t} \{ -\omega^2 L + (i\omega R + \frac{1}{C}) \}$$

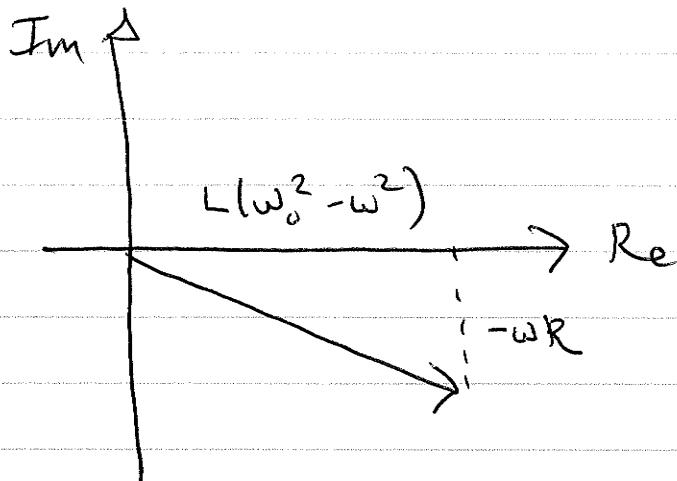
$$A = \frac{V}{\frac{1}{C} - \omega^2 L + i\omega R}$$

$$= \frac{V}{L(\frac{1}{C} - \omega^2) + i\omega R} \Rightarrow \frac{V}{L(\omega_0^2 - \omega^2) + i\omega R}$$

$$= \frac{V}{L^2(\omega_0^2 - \omega^2)^2 + \omega^2 R^2} (L(\omega_0^2 - \omega^2) - i\omega R)$$

C8//

$$L^*(\omega_0^2 - \omega^2) = i\omega R = \sqrt{L^2(\omega_0^2 - \omega^2)^2 + \omega^2 R^2} e^{-i\phi}$$



$$\tan \phi = \frac{\omega R}{L(\omega_0^2 - \omega^2)}$$

$$A = \frac{V}{\sqrt{L^2(\omega_0^2 - \omega^2)^2 + \omega^2 R^2}} e^{-i\phi}$$

$$Q(t) = \frac{V}{\sqrt{L^2(\omega_0^2 - \omega^2)^2 + \omega^2 R^2}} e^{i(\omega t - \phi)}$$

what can you say in words about this soln?

Eg "Resonance" if $\omega = \omega_0$; $Q(t)$ is big

$Q(t)$ "lags" $V(t)$ by angle ϕ :

"Impedance" is generalization of resistance $I = \frac{dQ}{dt}$

$$\frac{V}{\sqrt{R^2 + L^2 \left(\frac{\omega_0^2}{\omega^2} - 1 \right)}} \leftarrow \frac{V}{R}$$

C9//

summary Complex #'s are very useful (like rational #'s). Need to understand rules for manipulating them (like rules for adding/multiplying/dividing "fractions").