PHYSICS 104A, FALL 2015
MATHEMATICAL PHYSICS

Assignment Seven C, Due Monday, December 5, 5:00 pm.

[1.] The displacement from equilibrium of a violin string of length L is given by $y(x,t)$. The string is plucked so that its initial displacement is

$$y(x,t=0) = \frac{2h}{L}\,x \qquad\qquad 0 < x < \frac{L}{2}$$

$$y(x,t=0) = \frac{2h}{L}\,(L-x) \qquad\qquad \frac{L}{2} < x < L$$

It is released from rest so that the initial velocity $\partial y(x,t)/\partial t\big|_{t=0} = 0$. Compute $y(x,t)$ for later times. (A considerable amount of the mathematics will be similar to problem number one.)

[2.] Solve Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

inside the rectangular box $0 < x < L$ and $0 < y < H$. The edges of the box along the axes are grounded: $V(x=0,y) = 0$ and $V(x,y=0) = 0$, as is the potential along the top edge: $V(x,y=H) = 0$. Along the right edge $x = L$ the potential rises linearly from $V(L,y=0) = 0$ to $V(L,y=H) = V_0$. That is, $V(L,y) = V_0 y/H$.

[3.] A quantum mechanical particle has initial wave function $\psi(x,0) = ne^{-a|x|/2}$. What value of n is needed for normalization? What is the distribution of momentum $c(k)$? Check that $c(k)$ is normalized. Compute $\Delta x \Delta p$. Does it have the minimum possible value $\hbar/2$ as did the Gaussian wave function discussed in class?

1-1

Physics 104A
Assignment 7

The wave eqn is $\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$

where $y(xt)$ is the displacement from equilibrium.

For a vibrating string $y(0,t) = y(L,t) = 0$.

If we separate variables $y(xt) = f(x)g(t)$

we see that $f''(x) = -k^2 f(x)$

$g''(t) = -v^2k^2 g(t)$

so that $f(x) = \sin kx \,;\, \cos kx$

$g(t) = \sin wt \,;\, \cos wt$ where $w = vk$

The boundary conditions eliminate the $\cos kx$ soln

and require $k = n\pi/L$ so that

$$y(x,t) = \sum_1^\infty \sin\frac{n\pi x}{L}\left\{a_n \cos\frac{n\pi v t}{L} + b_n \sin\frac{n\pi v t}{L}\right\}$$

and $$\frac{\partial y}{\partial t}(x,t) = \sum_1^\infty \left(\sin\frac{n\pi x}{L}\right)\left(\frac{n\pi v}{L}\right)\left\{-a_n \sin\frac{n\pi v t}{L} + b_n \cos\frac{n\pi v t}{L}\right\}$$

1-2

In this particular problem we are told

$$\frac{\partial y}{\partial t}(x,0) = 0 \qquad \text{which implies } b_n = 0$$

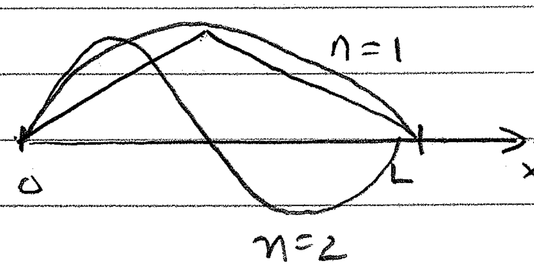
$$\text{so} \qquad y(x\,t) = \sum_1^{\infty} a_n \sin\frac{n\pi x}{L}\cos\frac{n\pi v t}{L}$$

we also know

$$y(x,0) = \sum_1^{\infty} a_n \sin\frac{n\pi x}{L} = \begin{cases} 2h/_L\, x & 0<x<L/2 \\ 2h/_L(L-x) & L/2<x<L \end{cases}$$

The final step is getting the a_n. Multiply by $\sin\frac{m\pi x}{L}$ and integrate

using orthogonality: $\frac{L}{2} a_m = \int_0^L y(x,0)\sin\frac{m\pi x}{L}\,dx$



← clearly get Ø for n even and for n odd $\int_0^{L/2} dx$ and $\int_{L/2}^{L} dx$ are same

$$L/2\; a_m = 2\int_0^{L/2} \frac{2h}{L} x \sin\frac{m\pi x}{L}\,dx \qquad m \text{ odd}$$

$$a_m = \frac{4}{L}\frac{2h}{L}\int_0^{L/2} \underbrace{x}_{u} \underbrace{\sin\frac{m\pi x}{L}\,dx}_{dv}$$

$$-\frac{L}{m\pi}x\cos\frac{m\pi x}{L}\Big|_0^{L/2} + \int_0^{L/x}\frac{L}{m\pi}\cos\frac{m\pi x}{L}\,dx$$

1-3

$$a_m = \frac{8h}{L^2}\,\frac{L}{m\pi}\left\{-\frac{L}{2}\cos\frac{m\pi}{2} + 0 + \frac{L}{m\pi}\sin\frac{m\pi x}{L}\Big|_0^{L/2}\right\}$$

$$= \frac{8h}{m\pi L}\left\{-\frac{L}{2}\cos\frac{m\pi}{2} + \frac{L}{m\pi}\sin\frac{m\pi}{2}\right\}$$

$$= -\frac{8h}{2m\pi}\cos\frac{m\pi}{2} + \frac{8h}{m^2\pi^2}\sin\frac{m\pi}{2}$$

recall m is odd so $\cos\frac{m\pi}{2} = 0$ and $\sin\frac{m\pi}{2} = (-1)^{\frac{m-1}{2}}$

$$\boxed{\begin{aligned} a_m &= {}^{8h}\!/_{m^2\pi^2}\,(-1)^{\frac{m-1}{2}} \\ y(x,t) &= \sum_{m=\text{odd}} a_m \sin\frac{m\pi x}{L}\cos\frac{m\pi v t}{L} \end{aligned}}$$

To avoid funny looking $m-1/2$ sometimes we write odd integers as $2m+1$ $m=0,1,2,..$ and then

$$a_m = {}^{8h}\!/_{(2m+1)^2\pi^2}\,(-1)^m$$

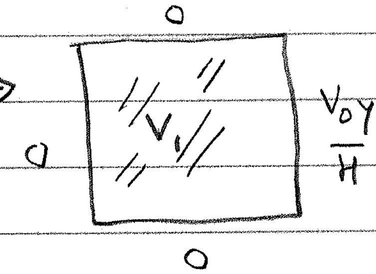
$$y(x,t) = \sum_{m=0,1,2} a_m \sin\frac{(2m+1)\pi x}{L}\cos\frac{(2m+1)\pi x}{L}$$

of course that makes arguments of sine, cosine more complicated...

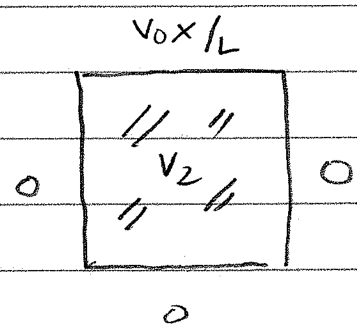
2-1

Because V obeys a linear eqn we can solve

for V_1 with bdy conditions $\Longrightarrow$



and V_2 with bdy conditions



then $V = V_1 + V_2$

Separating variables for V_1 we get $V_1 = f(x) g(y)$

$$-\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 g}{\partial y^2} = -k^2$$

Thus $g(y) = \sin ky$, $\cos ky$ $\qquad f(x) = \sinh kx$, $\cosh ky$

bdy conditions $\Rightarrow$ no $\cos ky$ soln and also $k = \frac{n\pi}{H}$

bdy conditions $\Rightarrow$ no $\cosh kx$ soln

$$V_1(xy) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi Y}{H} \cosh \frac{n\pi x}{H}$$

Obviously $V_2(x,y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \cosh \frac{n\pi Y}{L}$

To determine a_n set $x = L$

$$V_0 Y/H = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi Y}{H} \cosh\frac{n\pi L}{H}$$

Integrate $\sin m\pi Y/H$ on both sides, use orthogonality

$$\int_0^H \frac{V_0 Y}{H} \sin\frac{m\pi Y}{H}\, dy = \frac{H}{2} a_m \cosh\frac{m\pi L}{H}$$

We did this integral in problem #1, .., but ... again

$$\frac{V_0}{H}\int_0^H \underset{u}{Y} \underbrace{\sin\frac{m\pi Y}{H}\, dY}_{dv} = \frac{V_0}{H}\frac{H}{m\pi}\left\{ -Y\cos\frac{m\pi Y}{H}\Big|_0^H + \int_0^H \cos\frac{m\pi Y}{H}\, dy \right\}$$

$$= \frac{V_0}{m\pi}\left\{ H\cos m\pi + \frac{H}{m\pi}\sin\frac{m\pi Y}{H}\Big|_0^H \right\}$$

$$= \frac{V_0}{m\pi} H \left\{ (-1)^{m+1} \right\}$$

$$a_m = \frac{2V_0}{m\pi}(-1)^{m+1}\, 1/\cosh m\pi L/H$$

$$V(x,y) = \sum_1^{\infty} a_m \sin\frac{m\pi Y}{H}\cosh\frac{m\pi X}{H} \quad \leftarrow V_1$$

$$+ \sum_1^{\infty} b_m \sin\frac{m\pi X}{L}\cosh\frac{m\pi Y}{L} \quad \leftarrow V_2$$

$$b_m = \frac{2V_0}{m\pi}(-1)^{m+1}\, 1/\cosh m\pi H/L$$

just exchange

$X \leftrightarrow Y$

$L \leftrightarrow H$

1-1

$$\psi(x,0) = n e^{-a|x|/2}$$

Normalization condition:

$$1 = \int_{-\infty}^{\infty} |\psi(x,0)|^2 dx = n^2 \int_{-\infty}^{\infty} e^{-a|x|} dx$$

$$= 2n^2 \int_0^{\infty} e^{-ax} dx = 2n^2 \left. \frac{e^{-ax}}{-a} \right|_0^{\infty} = \frac{2n^2}{a}$$

$$\therefore \quad n = \sqrt{a/2}$$

Momentum prob dist is fourier transform

$$\phi(p,0) = \int_{-\infty}^{\infty} n e^{-a|x|/2} e^{ikx} dx \frac{1}{\sqrt{2\pi}} \quad \text{with } k = p/\hbar$$

$$= \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{ax/2} e^{ikx} dx + \frac{n}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax/2} e^{ikx} dx$$

$$= \frac{n}{\sqrt{2\pi}} \left. \frac{e^{(a/2+ik)x}}{a/2+ik} \right|_{-\infty}^{0} + \frac{n}{\sqrt{2\pi}} \left. \frac{e^{(-a/2+ik)x}}{-a/2+ik} \right|_0^{\infty}$$

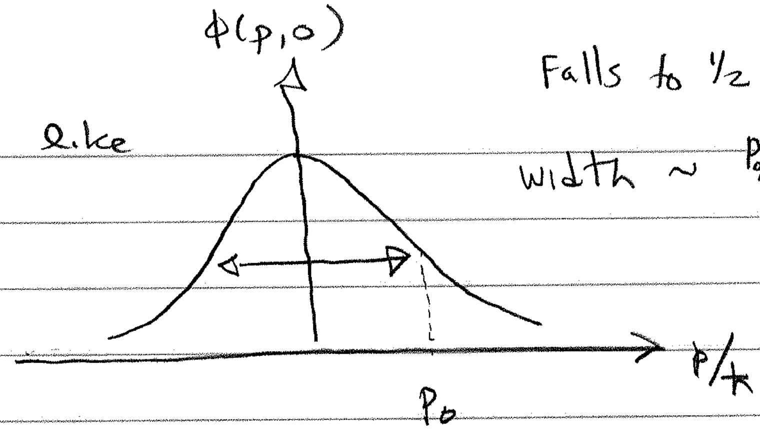
$$= \frac{n}{\sqrt{2\pi}} \frac{+1}{a/2+ik} + \frac{n}{\sqrt{2\pi}} \frac{-1}{-a/2+ik}$$

$$= \frac{n}{\sqrt{2\pi}} \frac{-a/2 + ik - a/2 - ik}{-a^2/4 - k^2} = \frac{na}{\sqrt{2\pi}} \Big/ (a^2/4 + k^2)$$

$$\phi(p,0) = \sqrt{\frac{a}{4\pi}} \frac{a}{\frac{a^2}{4} + p^2/\hbar^2}$$

1-2

This looks like

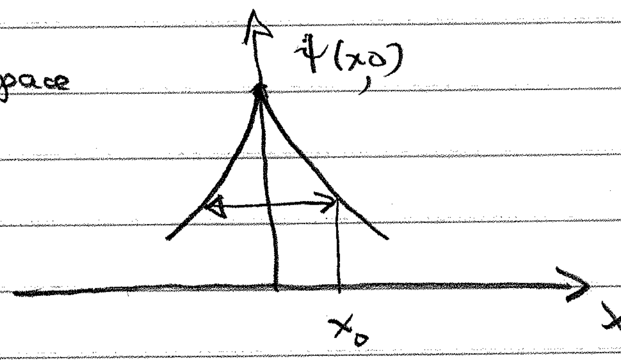


Falls to $1/2$ its $p=0$ value at

width $\sim$ $P_0/\hbar = a/2$

$P_0 = \hbar a/2$

Meanwhile in real space



Falls to $1/e$ of its value when $x_0 = 2/a$

So exhibits uncertainty principle: As $\psi(x,0)$ gets more narrow $\phi(p,0)$ gets broader and vice versa

Check normalization

$$1 \stackrel{?}{=} \int_{-\infty}^{\infty} |\phi(p,0)|^2 \frac{dp}{\hbar} = \frac{a}{4\pi} a^2 \int_{-\infty}^{\infty} \frac{dp/\hbar}{\left(\frac{a^2}{4} + \frac{p^2}{\hbar^2}\right)^2}$$

Let $p/\hbar = a/2 \tan\theta$ $\qquad p^2/\hbar^2 = a^2/4 \tan^2\theta$

$$dp = \frac{a\hbar}{2}\sec^2\theta\, d\theta$$

$$= \frac{a^3}{4\pi}\frac{2}{\hbar}\int_0^{\pi/2} \frac{a\hbar/2 \sec^2\theta\, d\theta}{(a^2/4)^2\left[1+\tan^2\theta\right]^2} = \frac{a^3}{2\pi}\frac{a\hbar}{2}\frac{16}{a^4}\frac{1}{\hbar}\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$= \frac{4}{\pi}\frac{1}{2}\frac{\pi}{2} = 1 \quad ✓✓$$

length of integration interval

$\cos^2\theta$ has average value $1/2$

Check uncertainty relation

$$\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2$$

$\langle x\rangle = 0$ by symmetry

$$\langle x^2\rangle = n^2 \int_{-\infty}^{\infty} x^2 e^{-a|x|} dx$$

$$= 2n^2 \int_0^{\infty} x^2 e^{-ax} dx$$

$$= 2n^2 \left\{ \frac{x^2 e^{-ax}}{-a}\Big|_0^{\infty} + \frac{2}{a}\int_0^{\infty} x e^{-ax} dx \right\}$$

$$= \frac{4n^2}{a} \left\{ \frac{x e^{-ax}}{-a}\Big|_0^{\infty} + \frac{1}{a}\int_0^{\infty} e^{-ax} dx \right\}$$

$$= \frac{4n^2}{a^2} \frac{e^{-ax}}{-a}\Big|_0^{\infty} = \frac{4n^2}{a^3} = \frac{4}{a^3}\frac{a}{2} = \frac{2}{a^2}$$

$$\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2$$

$\langle p\rangle = 0$ by symmetry

$$\langle p^2\rangle = \int_{-\infty}^{\infty} |\phi(p,0)|^2 p^2 \frac{dp}{\hbar} = \frac{a}{4\pi}\frac{2a^2}{\hbar}\int_0^{\infty} \frac{p^2}{\left(\frac{a^2}{4}+\frac{p^2}{\hbar^2}\right)^2} dp$$

Same trig substitution as before...

$$= \frac{a^3}{2\pi\hbar}\frac{a\hbar}{2}\frac{16}{a^4}\left(\frac{\hbar a}{2}\right)^2 \underbrace{\int_0^{\pi/2} \sin^2\theta \, d\theta}_{1/2 \, \pi/2 \text{ as before.}}$$

Thus $\langle p^2 \rangle = \frac{4}{\pi} \frac{\hbar^2 a^2}{4} \frac{\pi}{4} = \frac{\hbar^2 a^2}{4}$

In summary $\Delta x = \sqrt{2}/a$

$\Delta p = \hbar a/2$

$\Delta x \, \Delta p = \hbar/\sqrt{2}$ ← this is larger than the minimum possible value $\hbar/2$.