

PHYSICS 104A, FALL 2015
MATHEMATICAL PHYSICS

Assignment Six, Due Friday, November 13, 5:00 pm.

[1.] Compute the Fourier Series for the function $f(x) = 4 - x^2$ with $-2 < x < 2$ which repeats with period $L = 4$. Extra Credit: Plot the function and its Fourier series approximants ending the series at maximal $n = 1, 2, 4, 8, 16$.

[2.] An LRC circuit is driven by a voltage

$$\begin{aligned} V(t) &= -V_0 & -T < t < 0 \\ V(t) &= +V_0 & 0 < t < +T \end{aligned}$$

which is periodic with period $2T$. Find the charge on the capacitor $Q(t)$ as a function of time. You can leave your answer as an infinite sum. Also, ignore the 'transients', i.e. find the solution for $t \gg 2L/R$ so that the initial values $Q(t=0)$ and $I(t=0)$ are irrelevant.

[3.] A quantum mechanical particle is initially confined in the left hand half of an infinite square well $0 < x < L$. That is,

$$\begin{aligned} \psi(x, t=0) &= \sqrt{\frac{2}{L}} & 0 < x < \frac{L}{2} \\ \psi(x, t=0) &= 0 & \frac{L}{2} < x < L \end{aligned}$$

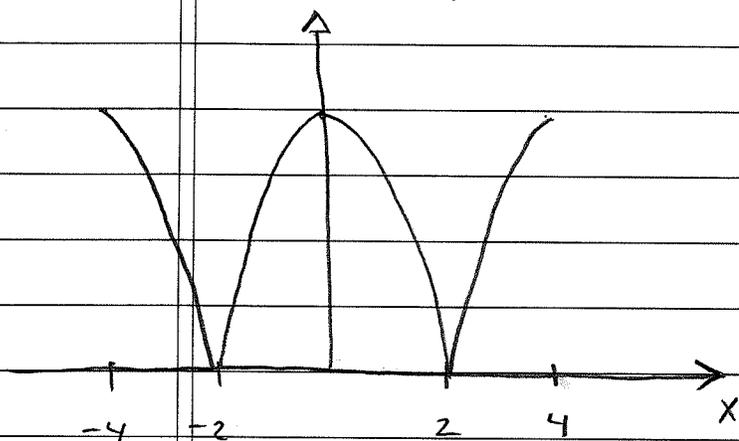
Compute the wave function $\psi(x, t)$ at later time t . Extra Credit: Plot the Fourier series approximant for $\psi(x, t=0)$ ending the series at maximal $n = 32, 128$. Extra Extra Credit: Plot $|\psi(x, t)|^2$ using maximal $n = 128$ for a couple of interesting t values. (The point is to watch the wave function spread out into the unoccupied region of the box.)

1-1

Physics 104A

Assignment 6

$$f(x) = 4 - x^2$$



We immediately see $f(x)$ will have only cosine terms since it is even, so $b_n = 0 \forall n$. Also, since $f(x)$ already looks a lot like an inverted cosine, we might expect the Fourier series to be dominated by its first term, i.e. a_n to fall rapidly with n .

$$L = 2$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

$$f(x) = a_0/2 + \sum_n a_n \cos \frac{n\pi x}{L}$$

$$\text{First get } a_0 = \frac{1}{2} \int_{-2}^2 (4 - x^2) dx$$

$$= \frac{1}{2} \left(4x - \frac{x^3}{3} \right) \Big|_{-2}^2 = 8 - \frac{8}{3} = \frac{16}{3}$$

$$\text{To get } n \neq 0 \quad a_n = \frac{1}{2} \int_{-2}^2 (4 - x^2) \cos \frac{n\pi x}{2} dx$$

The first term is easy (1) (2)

$$\textcircled{1} \rightarrow 2 \int_{-2}^2 \cos \frac{n\pi x}{2} dx = \frac{4}{n\pi} \sin \frac{n\pi x}{2} \Big|_{-2}^2$$

$$= \frac{4}{n\pi} [\sin n\pi - \sin(-n\pi)] = 0$$

1-2

To do the second term first figure out integral

$$\int x^2 \cos kx dx = \operatorname{Re} \int x^2 e^{ikx} dx$$

$$\int x^2 e^{ikx} dx = \underbrace{x^2}_{u} \underbrace{\frac{e^{ikx}}{ik}}_{dv} - \int \underbrace{2x}_{du} \underbrace{\frac{e^{ikx}}{ik}}_{v} dx$$

$$= \frac{x^2 e^{ikx}}{ik} + \frac{2i}{k} \int \underbrace{x}_{u} \underbrace{e^{ikx}}_{dv} dx$$

$$\frac{x e^{ikx}}{ik} - \int \frac{e^{ikx}}{ik} dx$$

$$\frac{x e^{ikx}}{ik} + \frac{i}{k} \frac{e^{ikx}}{ik}$$

All together $\int x^2 e^{ikx} dx = -\frac{ix^2}{k} e^{ikx} + \frac{2}{k^2} x e^{ikx} + \frac{2i}{k^3} e^{ikx}$

Hence $\int x^2 \cos kx dx = \operatorname{Re} \int x^2 e^{ikx} dx = \frac{x^2 \sin kx}{k} + \frac{2x \cos kx}{k^2} - \frac{2 \sin kx}{k^3}$

check by differentiation

$$\frac{2x \sin kx}{k} + x^2 \cos kx + \frac{2 \cos kx}{k^2} - \frac{2x \sin kx}{k} - \frac{2 \cos kx}{k^2} \quad \checkmark \checkmark$$

1-3

Using this result

$$a_n = -\frac{1}{2} \int_{-2}^2 x^2 \cos \frac{n\pi x}{2} dx$$

$$= -\frac{1}{2} \left\{ \frac{x^2 \sin \frac{n\pi x}{2}}{(n\pi/2)} + \frac{2x \cos \frac{n\pi x}{2}}{(n\pi/2)^2} - \frac{2 \sin \frac{n\pi x}{2}}{(n\pi/2)^3} \right\}_{-2}^2$$

The sine terms all go away since $\sin n\pi = \sin(-n\pi) = 0$

$$a_n = -2 \left(\cos n\pi + \cos(-n\pi) \right) \frac{4}{n^2 \pi^2}$$

$$= -16/n^2 \pi^2 (-1)^n = 16/n^2 \pi^2 (-1)^{n+1}$$

Let's check by sketching the a_0 and a_1 terms

$$f(x) \doteq \frac{8}{3} + \frac{16}{\pi^2} \cos \frac{\pi x}{2}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ 2.67 & 1.62 \end{array}$$

| | | |
|-----|--|-----------|
| x | $\frac{16}{3} + \frac{16}{\pi^2} \cos \frac{\pi x}{2}$ | $4 - x^2$ |
|-----|--|-----------|

| | | |
|------|------|------|
| 0.00 | 4.29 | 4.00 |
|------|------|------|

| | | |
|------|------|------|
| 0.25 | 4.16 | 3.94 |
|------|------|------|

| | | |
|------|------|------|
| 0.50 | 3.81 | 3.75 |
|------|------|------|

| | | |
|------|------|------|
| 0.75 | 3.29 | 3.44 |
|------|------|------|

| | | |
|------|------|------|
| 1.00 | 3.07 | 3.00 |
|------|------|------|

1.25

1.50

1.75

See plots on next page!

1-4
#include <stdio.h>
#include <math.h>

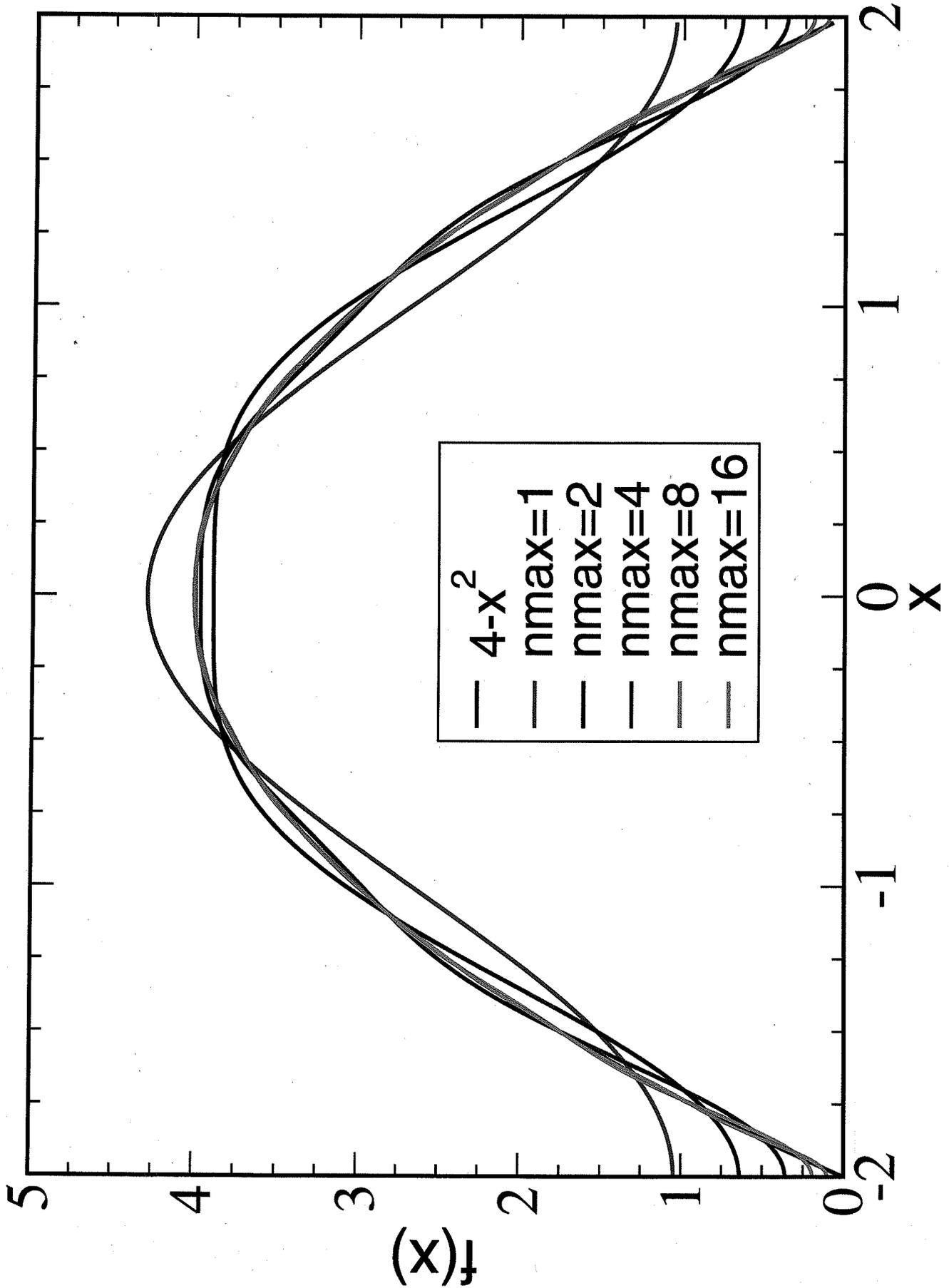
```
int main(void)
{
    FILE * fileout;
    fileout=fopen("fourier.dat","w");
    FILE * fileout2;
    fileout2=fopen("fourier2.dat","w");

    double dx,x,f,c,pi,sign,exact;
    int n,nmax,ix,nx;
    printf("\nEnter nmax");
    printf("\n");
    scanf("%i",&nmax);
    printf("Enter nx");
    printf("\n");
    scanf("%i",&nx);
    dx=2./nx;
    pi=4.*atan(1.);
    c=16./(pi*pi);

    for (ix=-nx; ix<nx+1; ix=ix+1)
    {
        f=8./3.;
        x=ix*dx;
        sign=1.;
        fprintf(fileout,"\n %12.4f",x);
        fprintf(fileout,"\n      (%4i %12.6f",0,f);

        for (n=1; n<nmax+1; n=n+1)
        {
            f=f+sign*c*cos(pi*n*x/2.)/n/n;
            fprintf(fileout,"\n      %4i %12.6f",n,f);
            sign=-sign;
        }
        exact=4.-x*x;
        fprintf(fileout,"\n      %12.6f",exact);
        fprintf(fileout2,"\n %12.6f %12.6f %12.6f",x,f,exact);
        fprintf(fileout,"\n ");
    }

    fclose(fileout);
    fclose(fileout2);
    return 0;
}
```



2-1

$Q(t)$ obeys Kirchoff's law

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = V(t)$$

For $V(t) = 0$ we try a soln $Q e^{i\omega t}$

$$-L\bar{\omega}^2 + iR\bar{\omega} + \frac{1}{C} = 0$$

$$\bar{\omega} = \left\{ -iR \pm \sqrt{-R^2 + 4L/C} \right\} / (-2L)$$

$$= \frac{iR}{2L} \pm \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

so that $Q(t) = e^{-Rt/2L} \left\{ A \cos \omega' t + B \sin \omega' t \right\}$

$$\text{with } \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Our driving voltage can be decomposed in a Fourier series. Because $V(t)$ is odd, only the sin terms are nonzero

$$V(t) = \sum a_n \sin n\pi t / T$$

$$a_n = \frac{1}{T} \int_{-T}^T V(t) \sin n\pi t / T dt$$

Anatoliy did this problem in his lecture for years

2-2

We will reverse

$$q_n = \frac{1}{T} \left\{ \int_{-T}^0 -V_0 \sin \frac{n\pi t}{T} dt + \int_0^T V_0 \sin \frac{n\pi t}{T} dt \right\}$$



these are equal

$$= -\frac{2V_0}{T} \frac{T}{n\pi} \cos \frac{n\pi t}{T} \Big|_0^T$$

$$= -\frac{2V_0}{n\pi} \{ (-1)^n - 1 \}$$

$$= \frac{2V_0}{n\pi} \{ 1 - (-1)^n \}$$

Vanishes for n even and for n odd $= \frac{4V_0}{n\pi}$

So we know
$$V(t) = \sum_{n \text{ odd}} \frac{4V_0}{n\pi} \sin \frac{n\pi t}{T}$$

We need response to a single one of these and

then we will combine since our differential eqn is linear

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = P \sin \omega t = \text{Im } P e^{i\omega t}$$

guess soln $Q e^{i\omega t}$ for $P e^{i\omega t}$

$$-\omega^2 L Q + i\omega R Q + \frac{1}{C} Q = P$$

$$Q = \frac{P}{(1/C - \omega^2 L) + i\omega R}$$

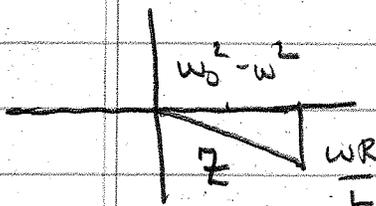
2-3

$$Q(t) = \text{Im} \frac{P}{(1/C - \omega^2 L) + i\omega R} e^{i\omega t}$$

$$\uparrow$$

$$\frac{P(1/C - \omega^2 L) - i\omega R}{(1/C - \omega^2 L)^2 + \omega^2 R^2}$$

Define $\omega_0^2 = 1/LC$



$$\frac{P(\omega_0^2 - \omega^2) - i\omega R/L}{(\omega_0^2 - \omega^2)^2 + \omega^2 R^2/L^2} = \frac{P e^{-i\phi}}{Z}$$

$$Z^2 = (\omega_0^2 - \omega^2)^2 + \omega^2 R^2/L^2 \quad \text{where} \quad \tan \phi = \omega R/L / Z$$

$$Q(t) = \frac{P}{Z} \sin(\omega t - \phi)$$

$$Z^2 = (\omega_0^2 - \omega^2)^2 + \omega^2 R^2/L^2$$

is solution for
 $P \sin \omega t$.

$$\tan \phi = \omega R/L / Z$$

$$Q(t) = e^{-Rt/2L} \left\{ A \cos \omega' t + B \sin \omega' t \right\} \quad \leftarrow A, B \text{ determined by } Q(0), I(0)$$

$$+ \sum_{n \text{ odd}} \frac{4V_0}{n\pi} \frac{1}{Z_n} \sin\left(\frac{n\pi t}{L} - \phi_n\right)$$

$$\omega' = \sqrt{\omega_0^2 - R^2/4L^2}; \quad \omega_0^2 = 1/LC; \quad Z_n^2 = \left(\omega_0^2 - \frac{n^2\pi^2}{L^2}\right)^2 + \frac{n^2\pi^2 R^2}{L^2}$$

$$\tan \phi_n = n\pi R/L^2$$

3-1

To solve the Schrödinger Eqn we

① Find eigenfunctions $H \phi_n = E_n \phi_n$

② Expand $\psi(x, t=0)$ in $\phi_n(x)$: $\psi(x, 0) = \sum a_n \phi_n(x)$

③ $\psi(x, t) = \sum a_n \phi_n(x) e^{-iE_n t / \hbar}$

For an infinite square well $\phi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$

as discussed in class. So we need an for our

particular $\psi(x, 0) = \begin{cases} \sqrt{\frac{2}{L}} & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases}$

In general $a_n = \int_0^L \psi(x, 0) \phi_n(x) dx$
 $= \int_0^L \psi(x, 0) \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} dx$

for our $\psi(x, 0) \rightarrow \int_0^{L/2} \frac{2}{L} \sin \frac{n\pi x}{L} dx$

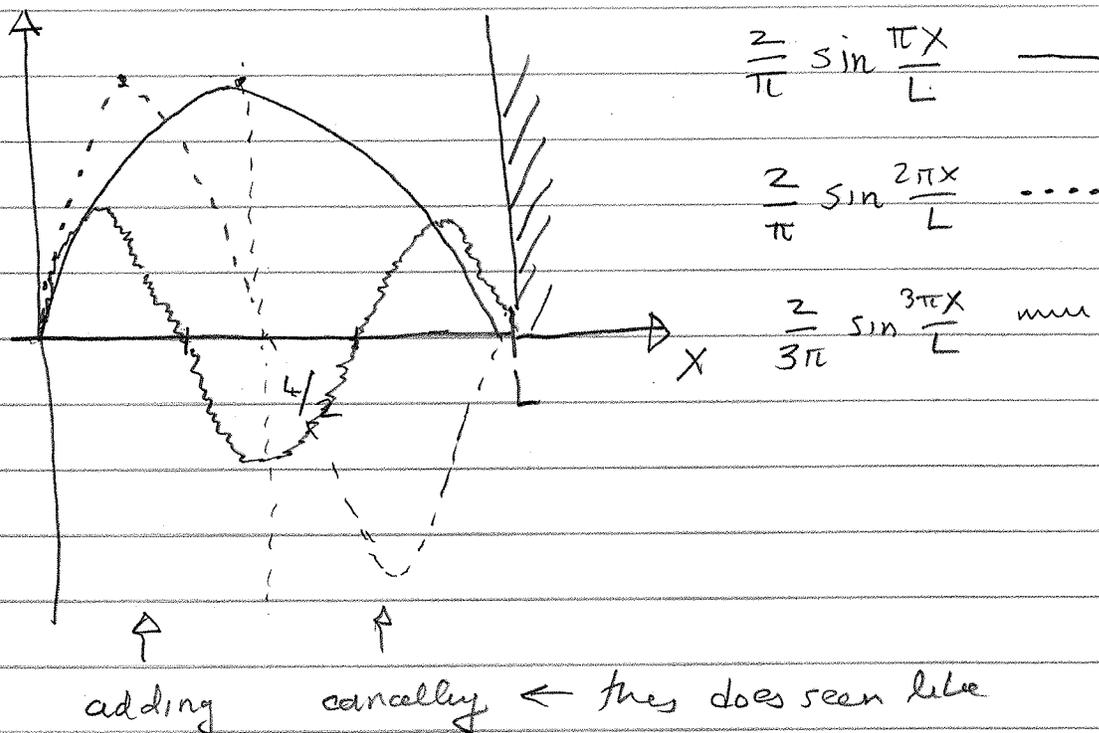
$$= -\frac{2}{L} \frac{L}{n\pi} \cos \frac{n\pi x}{L} \Big|_0^{L/2}$$

$$= -\frac{2}{n\pi} \left\{ \cos \frac{n\pi}{2} - 1 \right\} = \frac{2}{n\pi} \left\{ 1 - \cos \left(\frac{n\pi}{2} \right) \right\}$$

for the first few n $a_1 = \frac{2}{\pi}$ $a_2 = \frac{2}{2\pi}$ $a_3 = \frac{2}{3\pi}$ $a_4 = 0$

3-2

Do these values make sense? Try a sketch



So our soln is

$$\psi(x,t) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} e^{-i \frac{n^2 \pi^2 \hbar^2}{L^2 2m} t / \hbar}$$

with $a_n = \frac{2}{n\pi} \left\{ 1 - \cos\left(\frac{n\pi}{2}\right) \right\}$.

3-3

```
#include <stdio.h>
#include <math.h>

int main(void)
{
    FILE * fileout;
    fileout=fopen("fourier.dat","w");
    FILE * fileout2;
    fileout2=fopen("fourier2.dat","w");

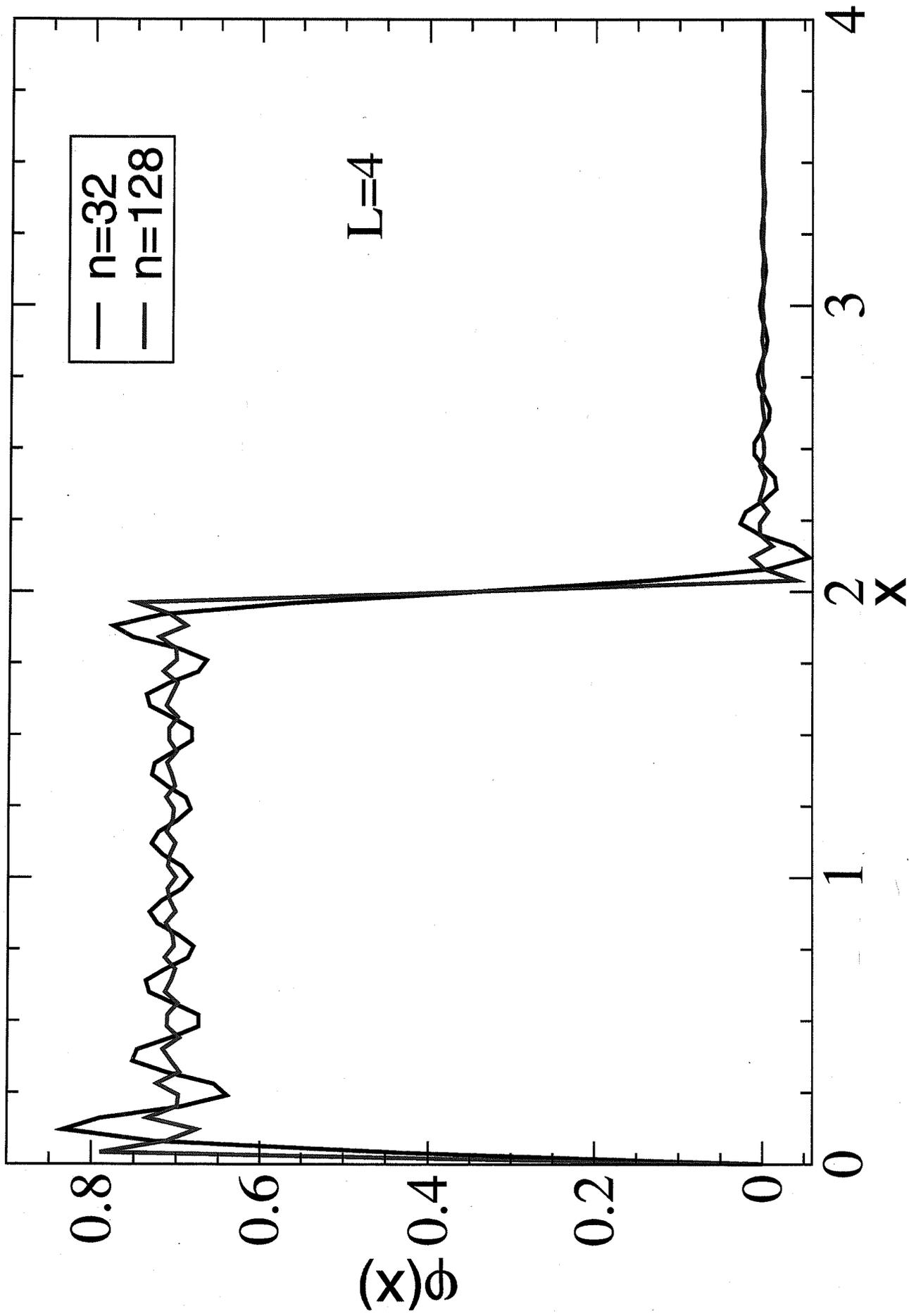
    double L,dx,x,f,pi,an;
    int n,nmax,ix,nx;
    printf("\nEnter nmax");
    printf("\n");
    scanf("%i",&nmax);
    printf("Enter nx");
    printf("\n");
    scanf("%i",&nx);
    L=4.;
    dx=L/nx;
    pi=4.*atan(1.);

    for (ix=0; ix<nx+1; ix=ix+1)
    {
        f=0.;
        x=ix*dx;
        fprintf(fileout,"\n %12.4f",x);
        fprintf(fileout,"\n      %4i %12.6f",0,f);

        for (n=1; n<nmax+1; n=n+1)
        {
            an=2.*( 1.-cos(n*pi/2) )/n/pi;
            f=f+an*sqrt(2./L)*sin(pi*n*x/L);
            fprintf(fileout,"\n      %4i %12.6f",n,f);
        }
        fprintf(fileout2,"\n %12.6f %12.6f",x,f);
        fprintf(fileout,"\n ");
    }

    fclose(fileout);
    fclose(fileout2);
    return 0;
}
```

3-4



3-5

```
#include <stdio.h>
#include <math.h>

int main(void)
{
    FILE * fileout;
    fileout=fopen("fourier.dat","w");
    FILE * fileout2;
    fileout2=fopen("fourier2.dat","w");

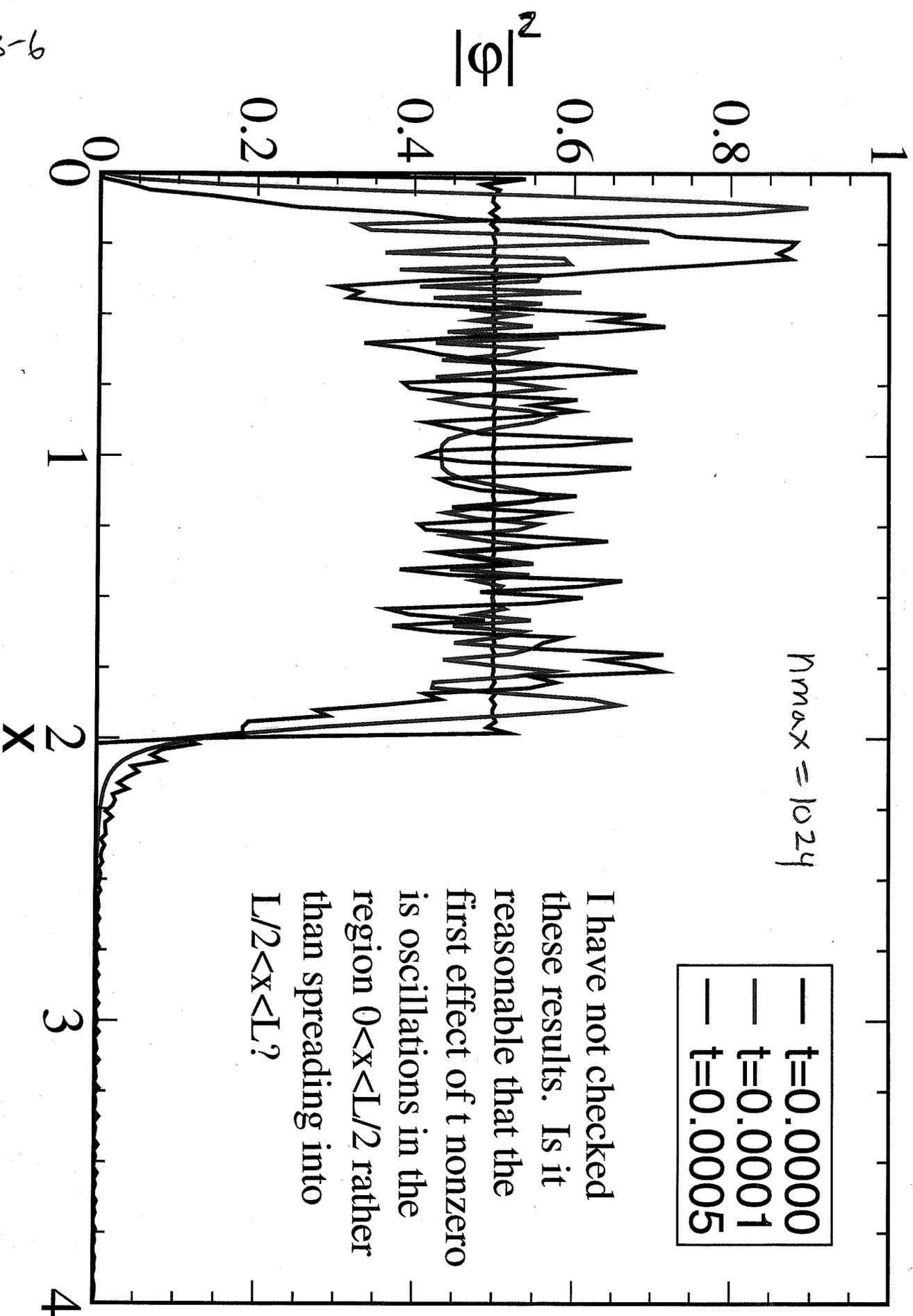
    double L,dx,x,fr,fi,f2,pi,an,t,ph;
    int n,nmax,ix,nx;
    printf("\nEnter nmax");
    printf("\n");
    scanf("%i",&nmax);
    printf("Enter nx");
    printf("\n");
    scanf("%i",&nx);
    printf("Enter t in units of 2 m L^2 / hbar");
    printf("\n");
    scanf("%lf",&t);

    L=4.;
    dx=L/nx;
    pi=4.*atan(1.);

    for (ix=0; ix<nx+1; ix=ix+1)
    {
        fr=0.;
        fi=0.;
        f2=fr*fr+fi*fi;
        x=ix*dx;
        fprintf(fileout,"\n %12.4f",x);
        fprintf(fileout,"\n      %4i %12.6f",0,f2);

        for (n=1; n<nmax+1; n=n+1)
        {
            ph=n*n*pi*pi*t;
            an=2.*( 1.-cos(n*pi/2) )/n/pi;
            fr=fr+an*cos(ph)*sqrt(2./L)*sin(pi*n*x/L);
            fi=fi+an*sin(ph)*sqrt(2./L)*sin(pi*n*x/L);
            f2=fr*fr+fi*fi;
            fprintf(fileout,"\n      %4i %12.6f",n,f2);
        }
        fprintf(fileout2,"\n %12.6f %12.6f",x,f2);
        fprintf(fileout,"\n ");
    }

    fclose(fileout);
    fclose(fileout2);
    return 0;
}
```



I have not checked these results. Is it reasonable that the first effect of t nonzero is oscillations in the region $0 < x < L/2$ rather than spreading into $L/2 < x < L$?