

PHYSICS 104A, FALL 2015
MATHEMATICAL PHYSICS

Assignment Five, Due Tuesday, November 3, 5:00 pm.

[1.] In the "usual" basis

$$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

a vector \vec{v} has the components given below, and an operator \mathcal{O} is represented by the matrix also given below.

$$\vec{v} \text{ ``=} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \quad \mathcal{O} \text{ ``=} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

What do the components of \vec{v} and the matrix representing \mathcal{O} become in the new basis:

$$\hat{f}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \hat{f}_2 = \begin{pmatrix} 0 \\ 0.6 \\ 0.8 \end{pmatrix} \quad \hat{f}_3 = \begin{pmatrix} 0 \\ -0.8 \\ 0.6 \end{pmatrix}$$

You can use the formulae from class, but it might be even better to rederive them yourself so they really stick with you.

[2.] The Hamiltonian of a quantum mechanical system, and its initial wavefunction, are given by

$$H = \hbar \omega_0 \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad \vec{\Psi}(t=0) = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}$$

Here ω_0 is a number with the units of inverse time (frequency). Using the basic rule for time evolution in quantum mechanics, $\vec{\Psi}(t) = \exp(-iHt/\hbar) \vec{\Psi}(t=0)$, compute $\vec{\Psi}(t)$.

[3.] For the same quantum mechanical system of problem 2, there is also a quantity Q you want to measure. It has the matrix

$$Q = Q_0 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

What are the probabilities to get different possible values for Q ? (Note: the answers will depend on time!)

1-1

Physics 104 A

Homework 5

$$\hat{f}_1 = \hat{e}_1$$

$$\hat{f}_2 = 0.6 \hat{e}_2 + 0.8 \hat{e}_3$$

$$\hat{f}_3 = -0.8 \hat{e}_2 + 0.6 \hat{e}_3$$

It is easy to see that

$$0.8 \hat{f}_2 + 0.6 \hat{f}_3 = \hat{e}_2$$

$$0.6 \hat{f}_2 - 0.8 \hat{f}_3 = \hat{e}_3$$

$$\text{Thus } \vec{v} = \hat{e}_1 + 2\hat{e}_2 - 3\hat{e}_3$$

$$= \hat{f}_1 + 2(0.6 \hat{f}_2 - 0.8 \hat{f}_3) - 3(0.8 \hat{f}_2 + 0.6 \hat{f}_3)$$

$$= \hat{f}_1 - 1.2 \hat{f}_2 - 3.2 \hat{f}_3$$

so in the $\{\hat{f}\}$ basis

$$\vec{v} \text{ "}= " \begin{pmatrix} 1 \\ -1.2 \\ -3.2 \end{pmatrix}$$

1-2

A check on this is $\|\vec{v}\|^2 = 1^2 + z^2 + (-3)^2 = 14$

and in the new bases $1^2 + (-1.2)^2 + (-3.4)^2$

$$= 1 + 1.44 + 11.56 = 14 \quad \checkmark$$

To transform $\hat{\mathbf{O}}$, recall we want

$$\vec{w}_{\text{new}} = \hat{\mathbf{O}}_{\text{new}} \vec{v}_{\text{new}}$$

$$\vec{w}_{\text{old}} = \hat{\mathbf{O}}_{\text{old}} \vec{v}_{\text{old}} \quad \downarrow \quad \text{use primes for } v_{\text{new}}$$

where $v'_1 \hat{f}_1 + v'_2 \hat{f}_2 + v'_3 \hat{f}_3$

$$= v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$$

$$v'_1 \hat{e}_1 + v'_2 (0.6 \hat{e}_2 + 0.8 \hat{e}_3) + v'_3 (-0.8 \hat{e}_2 + 0.6 \hat{e}_3)$$

$$= v'_1 \hat{e}_1 + (0.8 v'_2 - 0.8 v'_3) \hat{e}_2 + (0.8 v'_2 + 0.6 v'_3) \hat{e}_3$$

$$= v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$$

so $v'_1 = v_1$

$$0.6 v'_2 - 0.8 v'_3 = v_2 \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.6 & -0.8 \\ 0 & 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} v'_1 \\ v'_2 \\ v'_3 \end{pmatrix}$$

$$0.8 v'_2 + 0.6 v'_3 = v_3$$

\uparrow \uparrow \uparrow
old new
call this S

1.3

Notice that $S^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \\ 0 & -0.8 & 0.6 \end{pmatrix} = S^{-1}$

$$\vec{\omega}_{\text{old}} = \hat{\Omega}_{\text{old}} \vec{v}_{\text{old}}$$

$$S \vec{\omega}_{\text{new}} = \hat{\Omega}_{\text{old}} S \vec{v}_{\text{new}}$$

$$\vec{\omega}_{\text{new}} = S^T \hat{\Omega}_{\text{old}} \underbrace{S \vec{v}_{\text{new}}}$$

so this must equal $\vec{\omega}_{\text{new}}$

$$\hat{\Omega}_{\text{new}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \\ 0 & -0.8 & 0.6 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.6 & -0.8 \\ 0 & 0.8 & 0.6 \end{pmatrix}}$$

$$\begin{pmatrix} 0 & 0.6 & -0.8 \\ 1 & 0.8 & 0.6 \\ 0 & 0.6 & -0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0.6 & -0.8 \\ 0.6 & 0.96 & -0.28 \\ -0.8 & -0.28 & -0.96 \end{pmatrix}$$

Note $\text{Tr}[\hat{\Omega}_{\text{new}}] = \text{Tr}[\hat{\Omega}_{\text{old}}]$ is a simple check

we can make. It should also be true that

$$\det \hat{\Omega}_{\text{new}} = \det \hat{\Omega}_{\text{old}} = \phi.$$

1-4

Checking this

$$\det \hat{J}_{\text{new}} = -0.6 [-0.576 - 0.224] - 0.8 [-0.0168 + 0.768]$$
$$= 0.6 [0.8] - 0.8 [0.6] = \phi \quad \checkmark$$

2-1

First we need eigenvalues and eigenvectors of \hat{H}

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (4-\lambda)[(1-\lambda)^2 - 4] = 0$$

$$\lambda = 4 \quad 1-\lambda = \pm 2$$

$$\lambda = 4 \quad \lambda = -1, 3$$

eigenvectors $\lambda = 4 \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \hat{\phi}_1$

$$\lambda = 3 \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ +1 \\ 0 \end{pmatrix} = \hat{\phi}_2 \quad \lambda = -1 \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \hat{\phi}_3$$

(Actually, must multiply each λ by two)

Next we need to write $\vec{\psi}(t=0)$ as an

expansion in $\{\hat{\phi}\}$

$$\begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix} = a_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{a_2}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{a_3}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

By inspection $a_1 = 1/\sqrt{2}$

$$1/2 = (a_2 + a_3)/\sqrt{2}$$

$$1/2 = (a_2 - a_3)/\sqrt{2}$$

$$1 = a_2\sqrt{2} \quad a_2 = 1/\sqrt{2}$$

$$a_3 = 0$$

2-2

$$\hat{\psi}(t=0) = \frac{1}{\sqrt{2}} \hat{\phi}_1 + \frac{1}{\sqrt{2}} \hat{\phi}_2$$

$$\vec{\psi}(t) = e^{-i\hat{H}t} \hat{\psi}(t=0)$$

$$= e^{-i\hat{H}t} \left(\frac{1}{\sqrt{2}} \hat{\phi}_1 + \frac{1}{\sqrt{2}} \hat{\phi}_2 \right)$$

$$= e^{-i4\omega_0 t} \frac{1}{\sqrt{2}} \hat{\phi}_1 + e^{-i3\omega_0 t} \frac{1}{\sqrt{2}} \hat{\phi}_2$$

$$= e^{-i4\omega_0 t} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + e^{-i3\omega_0 t} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{\psi}(t) = \begin{pmatrix} \frac{1}{2} e^{-i3\omega_0 t} \\ \frac{1}{2} e^{-i3\omega_0 t} \\ \frac{1}{\sqrt{2}} e^{-i4\omega_0 t} \end{pmatrix}$$

Notice $\vec{\psi}^*(t) \vec{\psi}(t) = 1$ as required!

3-1

The possible values for measuring Q are given by its eigenvalues

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda) [(2-\lambda)(1-\lambda) - 1] + 1 [-1(1-\lambda)]$$

$$= (1-\lambda)[2 - 3\lambda + \lambda^2 - 2]$$

$$= (1-\lambda)\lambda(\lambda-3)$$

Eigenvalues $0, 1, 3 \leftarrow$ times Q_0 of course

Eigenvectors are (we did this problem in class!)

$$\hat{\phi}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\hat{\phi}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\hat{\phi}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

↑

$$\lambda = 0 \qquad \lambda = Q_0 \qquad \lambda = 3Q_0$$

3-2

For probabilities, expand $\vec{f}(t)$ in these
eigenvectors

$$\begin{pmatrix} \frac{1}{2} e^{-3iw_0 t} \\ \frac{1}{2} e^{-3iw_0 t} \\ \frac{1}{\sqrt{2}} e^{-4iw_0 t} \end{pmatrix} = a_1 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + a_2 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + a_3 \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

\Rightarrow
Invert this matrix!

Looks hard to invert, but because eigenvectors of Hermitian matrix are \perp the inverse of the matrix containing the eigenvectors is just the transpose (if eigenvectors real)

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} e^{-3iw_0 t} \\ \frac{1}{2} e^{-3iw_0 t} \\ \frac{1}{\sqrt{2}} e^{-4iw_0 t} \end{pmatrix}$$

$$a_1 = \frac{1}{\sqrt{3}} e^{-3iw_0 t} + \frac{1}{\sqrt{6}} e^{-4iw_0 t}$$

$$a_2 = \frac{1}{2\sqrt{2}} e^{-3iw_0 t} - \frac{1}{2} e^{-4iw_0 t}$$

$$a_3 = -\frac{1}{2\sqrt{6}} e^{-3iw_0 t} + \frac{1}{2\sqrt{3}} e^{-4iw_0 t}$$

3-3

$$P_1 = |q_1|^2 = q_1^* q_1 = \left(\frac{1}{\sqrt{3}} e^{3i\omega_0 t} + \frac{1}{\sqrt{6}} e^{4i\omega_0 t} \right) \left(\frac{1}{\sqrt{3}} e^{-3i\omega_0 t} + \frac{1}{\sqrt{6}} e^{-4i\omega_0 t} \right)$$

$$= \frac{1}{3} + \frac{1}{6} + \frac{1}{\sqrt{18}} (e^{-4i\omega_0 t} + e^{+i\omega_0 t})$$

\uparrow
 $2 \cos \omega_0 t$

$$= \frac{1}{2} + \frac{\sqrt{2}}{3} \cos \omega_0 t \quad \leftarrow \text{probability of measuring } \lambda_1 = 0$$

$$P_2 = |q_2|^2 = q_2^* q_2 = \left(\frac{1}{2\sqrt{2}} e^{+3i\omega_0 t} - \frac{1}{2} e^{4i\omega_0 t} \right) \left(\frac{1}{2\sqrt{2}} e^{-3i\omega_0 t} - \frac{1}{2} e^{-4i\omega_0 t} \right)$$

$$= \frac{1}{8} + \frac{1}{4} - \frac{1}{4\sqrt{2}} (e^{i\omega_0 t} + e^{-i\omega_0 t})$$

\uparrow
 $2 \cos \omega_0 t$

$$= \frac{3}{8} - \frac{\sqrt{2}}{4} \cos \omega_0 t \quad \leftarrow \text{prob of measuring } \lambda_2 = Q_0$$

$$P_3 = |q_3|^2 = q_3^* q_3 = \left(-\frac{1}{2\sqrt{6}} e^{+3i\omega_0 t} + \frac{1}{2\sqrt{3}} e^{+4i\omega_0 t} \right) \left(-\frac{1}{2\sqrt{6}} e^{-3i\omega_0 t} + \frac{1}{2\sqrt{3}} e^{-4i\omega_0 t} \right)$$

$$= \frac{1}{24} + \frac{1}{12} - \frac{1}{4\sqrt{18}} (e^{i\omega_0 t} + e^{-i\omega_0 t})$$

\uparrow
 $2 \cos \omega_0 t$

$$= \frac{1}{8} - \frac{\sqrt{2}}{12} \cos \omega_0 t \quad \leftarrow \text{prob of measuring } \lambda_3 = 3Q_0$$

Notice that $P_1 + P_2 + P_3 = 1$ as required!