5-1

$$E = mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \qquad (1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$$

$$= mc^2\left(1 + \left(-\tfrac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) + \frac{(-1/2)(-3/2)}{2}\left(\frac{-v^2}{c^2}\right)^2 + \dots\right)$$

$$= mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\,v^4/c^4 + \dots\right)$$

$$E = \underset{\substack{\text{rest}\\\text{energy}}}{mc^2} + \underset{\substack{\text{usual}\\\text{(classical)}\\\text{kinetic energy}}}{\tfrac{1}{2}mv^2} + \underbrace{\tfrac{3}{8}\,mv^2\left(v^2/c^2\right)^1}_{\substack{\text{first relativistic}\\\text{correction}}}$$

Solution, Assignment 4, Problem 2, Fall 2018

The general formula for the probability of $N-n$ heads and n tails when a coin is tossed N times is

$$\mathcal{P}(n) = \left(\begin{array}{c} N \\ n \end{array} \right) p^{N-n}\, q^{n}$$

Here p is the probability of heads in a single toss, and $q = 1-p$ is the probability of tails in a single toss.

For our problem with $N = 10, p = 0.6$, and $q = 0.4$:

$$\begin{aligned}
\mathcal{P}(10) &= \left(\begin{array}{c} 10 \\ 10 \end{array} \right) p^{10}\, q^{0} = 1(0.6)^{10}(0.4)^{0} && = 0.00605 \\
\mathcal{P}(9) &= \left(\begin{array}{c} 10 \\ 9 \end{array} \right) p^{9} q^{1} = 10\,(0.6)^{9}\,(0.4)^{1} && = 0.04031 \\
\mathcal{P}(8) &= \left(\begin{array}{c} 10 \\ 8 \end{array} \right) p^{8} q^{2} = 45\,(0.6)^{8}\,(0.4)^{2} && = 0.12093 \\
\mathcal{P}(7) &= \left(\begin{array}{c} 10 \\ 7 \end{array} \right) p^{7} q^{3} = 120\,(0.6)^{7}\,(0.4)^{3} && = 0.21499 \\
\mathcal{P}(6) &= \left(\begin{array}{c} 10 \\ 6 \end{array} \right) p^{6} q^{4} = 210\,(0.6)^{6}\,(0.4)^{4} && = 0.25082 \\
\mathcal{P}(5) &= \left(\begin{array}{c} 10 \\ 5 \end{array} \right) p^{5} q^{5} = 252\,(0.6)^{5}\,(0.4)^{5} && = 0.20065 \\
\mathcal{P}(4) &= \left(\begin{array}{c} 10 \\ 4 \end{array} \right) p^{4} q^{6} = 210\,(0.6)^{4}\,(0.4)^{6} && = 0.11148 \\
\mathcal{P}(3) &= \left(\begin{array}{c} 10 \\ 3 \end{array} \right) p^{3} q^{7} = 120\,(0.6)^{3}\,(0.4)^{7} && = 0.04247 \\
\mathcal{P}(2) &= \left(\begin{array}{c} 10 \\ 2 \end{array} \right) p^{2} q^{8} = 45\,(0.6)^{2}\,(0.4)^{8} && = 0.01062 \\
\mathcal{P}(1) &= \left(\begin{array}{c} 10 \\ 1 \end{array} \right) p^{1} q^{9} = 10\,(0.6)^{1}\,(0.4)^{9} && = 0.00157 \\
\mathcal{P}(0) &= \left(\begin{array}{c} 10 \\ 0 \end{array} \right) p^{0} q^{10} = 1\,(0.6)^{0}\,(0.4)^{10} && = 0.00010
\end{aligned}$$

The important think to notice is that the maximum probability is no longer when the numbers of heads and tails are equal. Instead it is when the number of heads is $N\,p$. Can you prove this?

3-1

REVIEW

prob of stepping right

In class we showed $\langle x \rangle = 0$ if $p = q = 1/2$

prob of stepping left

We did this by evaluating

$$\langle x \rangle = \sum_{n=0}^{N} (N-2n) \underbrace{\binom{N}{n} p^n q^{N-n}}$$

\# of steps to left

final location if n steps to left

prob of n steps to left.

We know

$$(p+q)^N = \sum \binom{N}{n} p^n q^{N-n}$$

and, differentiating $\partial/\partial p$

$$N(p+q)^{N-1} = \sum \binom{N}{n} n p^{n-1} q^{N-n}$$

$$Np(p+q)^{N-1} = \sum \binom{N}{n} n p^n q^{N-n}$$

Hence (using $p+q=1$)

$$\langle x \rangle = N - 2Np = N(1-2p) = 0 \text{ if } p = 1/2$$

3-2

The root mean square is

$$\sqrt{\langle x^2 \rangle} = \sqrt{\sum_{n=0}^{N} (N-2n)^2 \binom{N}{n} p^n q^{N-n}}$$

root mean square

$$N^2 - 4nN + 4n^2$$

We need $\sum n^2 \binom{N}{n} p^n q^{N-n}$

which we get by doing another derivative $\partial/\partial p$

$$N(N-1)\,p\,(p+q)^{N-2} + N(p+q)^{N-1}$$

$$= \sum \binom{N}{n} n^2 p^{n-1} q^{N-n}$$

so $$N(N-1)p^2(p+q)^{N-2} + Np(p+q)^{N-1} = \sum \binom{N}{n} n^2 p^n q^{N-n}$$

Thus (setting $p+q=1$)

$$\langle x^2 \rangle = N^2 - 4NNp + 4\{N(N-1)p^2 + Np\}$$

$$= N^2(1-4p+4p^2) - 4Np^2 + 4Np$$

$$= N^2(1-2p)^2 + 4N\underbrace{p(1-p)}_{1/4 \text{ if } p=1/2}$$

0 if $p=1/2$

$$\sqrt{\langle x^2 \rangle} = \sqrt{N}$$

Solution, Assignment 4, Problem 4, Fall 2018

The important principle here is that in a random process with two equally likely outcomes (a random walk, a coin toss, etc) of N steps, the expected distances one gets away from the origin are proportional to $\sqrt{N}$. The 'proportional to' takes into account the 'trivial' dependence on the size of the individual steps. For example if each step is 3 feet, then the distances away will be $3\sqrt{N}$.

So the 'fast' answer to this problem is that one expects the deviation from 5000 heads to be $\sqrt{10000} = 100$. The thinking is that 5000 heads and 5000 tails corresponds to a random walker ending up at the origin (heads=left; tails=right). So one should not be surprised if there are 5100 heads and 4900 tails, but 5300 heads and 4700 tails *would* be surprising!

As discussed in class, one can also get this result more rigorously by using Stirling's formula

$$\ln N! \sim N \ln N - N - 0.5 \ln (2\pi N)$$

Using the result

$$\mathcal{P}(n) = \binom{N}{n} \frac{1}{2^N}$$

we see that the ratio of the probability of 5100 heads to the probability of 5000 heads (the most likely outcome for a fair coin) is

$$\frac{\mathcal{P}(5100)}{\mathcal{P}(5000)} = \binom{10000}{5100} \binom{10000}{5000}^{-1} = \frac{5000!\; 5000!}{5100!\; 4900!}$$

Stirlings formula and some basic properties of logarithms and the symmetry of the binomial coefficients gives us the logarithm of this ratio,

$$\ln\left(\frac{\mathcal{P}(5100)}{\mathcal{P}(5000)}\right) = 2\,\big(\; 5000 \ln 5000 - 5000 - 0.5 \ln (10000\pi) - 5100 \ln 5100 - 5100 - 0.5 \ln (10200\pi)\;\big)$$

Here are some results:

```
P(5050)/P(5000) = 0.6066
P(5100)/P(5000) = 0.1353
P(5150)/P(5000) = 0.0111
P(5200)/P(5000) = 0.0003
```

So the probality of 5100 heads is still reasonly high, about 1/8 of the probability of 5000 heads. But 5150 heads is already pretty unlikely (one hundredth the probability of 5000 heads). 5200 heads is incredibly unlikely! **The $\sqrt{N}$ rule really does work!**

A little fortran code to do this:

```
c       GETS RATIO OF TWO BINOMIAL COEFFICIENTS (N,M1) / (N,M2)

        implicit none
        real*8 N,tpi,M1,M2,A,B1a,B1b,B2a,B2b
        real*8 logbin1,logbin2

        tpi=8.d0*datan(1.d0)
10      write (6,*) 'Enter N,M1,M2'
        read (5,*) N,M1,M2

        A  = N*dlog( N)- N+dlog(tpi* N)/2.d0

        B1a=M1*dlog(M1)-M1+dlog(tpi*M1)/2.d0
        M1=N-M1
        B1b=M1*dlog(M1)-M1+dlog(tpi*M1)/2.d0
        logbin1=A-B1a-B1b
        write (6,990) logbin1

        B2a=M2*dlog(M2)-M2+dlog(tpi*M2)/2.d0
        M2=N-M2
        B2b=M2*dlog(M2)-M2+dlog(tpi*M2)/2.d0
        logbin2=A-B2a-B2b
        write (6,991) logbin2

        write (6,992) logbin1-logbin2
        write (6,993) dexp(logbin1-logbin2)

990     format('log(binomial coefficient (N M1)=  ',f16.4)
991     format('log(binomial coefficient (N M2)=  ',f16.4)
992     format('difference                     =  ',f16.4)
993     format('exp(difference)                =  ',f16.4)

        go to 10
        end
```

Solution, Assignment 4, Problem 5, Fall 2018

The quick way to do this problem (but which is also a bit subtle) is to argue that the probability or having a boy and the probability of having a jirl are both 0.5000. There is no way that any scheme of society can change that basic biological fact. Therefore *no matter what rules for family outcomes are put in place the number of girls and boys will be equal*!!

This result is at the same time both *obvious* and *amazing*.

But let's prove it. A little thought will convince you that the possible families and their probabilities are:

```
G           1/2
BG          1/4
BBG         1/8
BBBG        1/16
...
```

We can check the probabilities add to one, as they must, by using the formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Choosing the value $x = 1/2$

$$\frac{1}{1-1/2} = 1 + 1/2 + 1/4 + 1/8 + 1/16 + \cdots$$

Since the left hand side is 2 we get

$$1 = 1/2 + 1/4 + 1/8 + 1/16 + \cdots$$

You will notice that every single family listed above has exactly one girl, so the expected (average) number of girls per family is one. But the number of boys varies. To get the expected (average) number of boys we must multiply the number of boys in each family times the probability of that family type:

$$\langle \# \text{ of boys} \rangle = (1/2)\,(0) + (1/4)\,(1) + (1/8)\,(2) + (1/16)\,(3) + \cdots$$

This is a sum we know how to do! If you differentiate the formula for $1/(1-x)$ above with respect to x you get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Multiplying by x^2 gives

$$\frac{x^2}{(1-x)^2} = x^2 + 2x^3 + 3x^4 + 4x^5 + \cdots$$

But if you set $x = 1/2$ the right hand side is the sum needed for the expected number of boys. So we conclude

$$\langle \,\#\text{ of boys}\, \rangle = \frac{(1/2)^2}{(1-(1/2))^2} = 1$$

So each familiy also has, on average, one boy. The numbers of boys and girls are equal.

If you want to have some fun, design your own society, for example families always want to have at least one girl and two boys. Compute the expected numbers of boys and girls. They will be equal. An interesting question is what the expected family size is!

$\boxed{6,7}$ * See figures, next pages, for $\tanh Jm/T$

and $T = \{2.0, 1.3, 0.7\} J$

$\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$

$= \left(1 + x + x^2/2 + x^3/6 - (1 - x + x^2/2 - x^3/6)\right)/(\ \ +\ \ )$

$= (2x + x^3/3)/(2 + x^2)$

$\cancel{= (2x + x^3/3)/}$

$= (x + x^3/6)/(1 + x^2/2)$

$= (x + x^3/6)(1 - x^2/2) = x + x^3/6 - x^3/2$

$= x - x^3/3$

Thus $\tanh \frac{Jm}{T} = \frac{J}{T} m - \frac{1}{3}\left(\frac{Jm}{T}\right)^3 + \ldots$

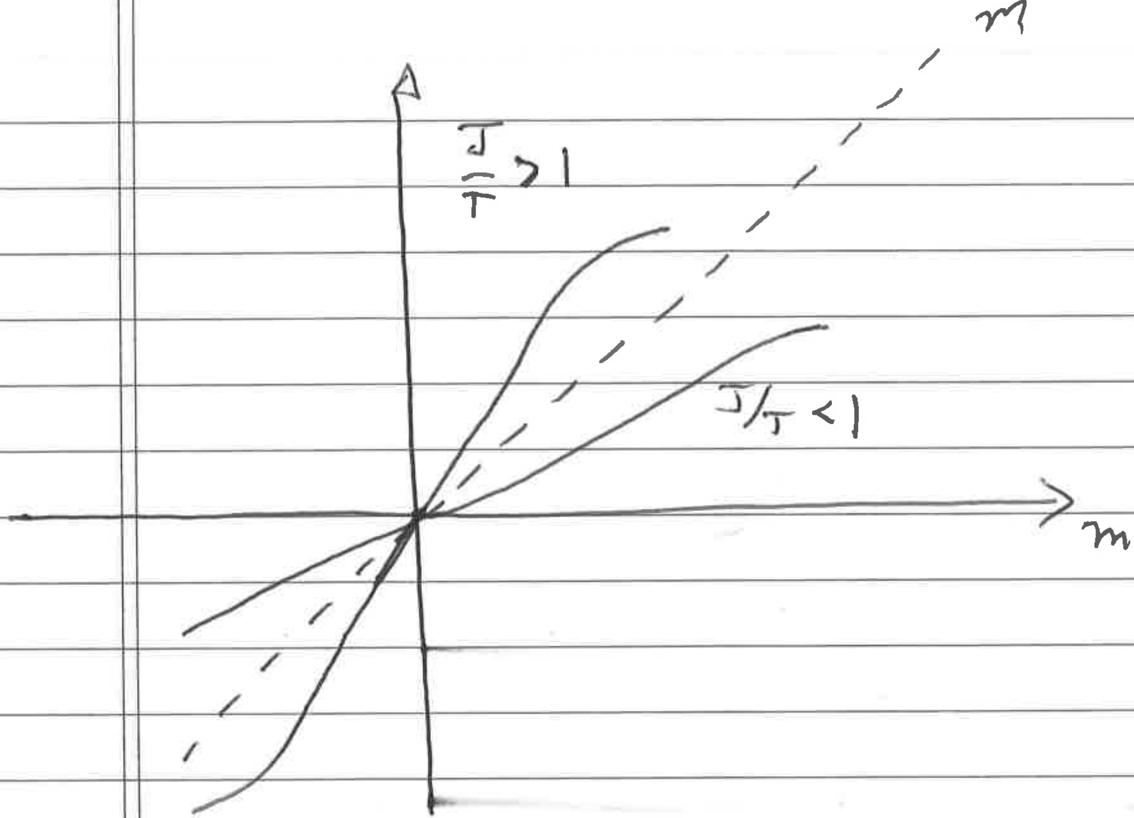
for small m this increases with slope J/T.

If $J/T < 1$ rises more slowly than m

and only intersects at $m = 0$.

If $J/T > 1$ rises faster than m then turns over and get $m \neq 0$ soln.

6;7 - 2



This is more clear on computer-drawn figures

Point of exercise:

Analysis of behavior of $\tanh x$ near $x = 0$

required to compute T_c ...