

PYHYSICS 104A, FALL 2016
MATHEMATICAL PHYSICS

Assignment Four, Due Friday, October 21, 5:00 pm.

[1.] In understanding the behavior of spin-1/2 particles in quantum mechanics, you will encounter the 2×2 ‘Pauli matrices’

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Compute the inverses of σ_x , σ_y , and σ_z .

[2.] The ‘commutator’ of two matrices A and B is symbolized by $[A, B]$ and is defined by

$$[A, B] = AB - BA$$

Show that the ‘spin’ matrices

$$S_x = \frac{\hbar}{2}\sigma_x \quad S_y = \frac{\hbar}{2}\sigma_y \quad S_z = \frac{\hbar}{2}\sigma_z$$

obey

$$[S_x, S_y] = i\hbar S_z \quad [S_y, S_z] = i\hbar S_x \quad [S_z, S_x] = i\hbar S_y$$

[3.] to understand how spin wave functions evolve in time t you will need the exponentials of the Pauli matrices,

$$A = e^{-it\sigma_x} \quad B = e^{-it\sigma_y} \quad C = e^{-it\sigma_z}$$

Using the definition

$$e^M = I + M + \frac{1}{2}M^2 + \frac{1}{6}M^3 + \dots$$

compute A , B and C .

[4.] In understanding the behavior of spin-1 particles in quantum mechanics, you will encounter the 3×3 matrices

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

What are the possible values you could get if you measure the x , y or z component of spin of a spin-one particle in an experiment? If your system is in the state $|\psi\rangle = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, what are the probabilities of measuring the different possible values of S_x ?

[5.] Construct the matrix $S^2 = S_x^2 + S_y^2 + S_z^2$ for the spin of a spin-one particle. What possible values can you get if you measure the square of the spin?

[6.] Go back to the spin-1/2 matrices of problems 1-3 and similarly construct the matrix $S^2 = S_x^2 + S_y^2 + S_z^2$. What possible values can you get if you measure the square of the spin?

[7.] Compute the commutators $[S_x, S_y]$, $[S_y, S_z]$, and $[S_z, S_x]$ for spin-1. How do the results compare to the spin-1/2 case?

Note: It's a bit weird that when you measure S^2 it is not really what you would naively expect. The reason is that S_x , S_y and S_z do not commute, and so you cannot measure them at the same time. Thus you cannot really hope to get S^2 by just taking the components, squaring them, and adding.

[8.] Extra Credit: You have (above) the matrices S^2 for $S = 1/2$ and for $S = 1$. What is the form of the matrix for the S^2 for a particle of general spin S ? This same result will come up in Physics 115AB when you figure out the eigenvalues of the orbital angular momentum of an electron going around a proton in the Hydrogen atom, since the angular momentum matrices obey the same basic rules as spin matrices. You saw it here first.

1-1

Since $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$ (see problem 3)

we know that $(\sigma_x)^{-1} = \sigma_x$; $(\sigma_y)^{-1} = \sigma_y$; and $(\sigma_z)^{-1} = \sigma_z$

If you want you can prove this also in the "usual" way

$$\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{\text{swap rows}} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right)$$

$\sigma_x \quad I \qquad I \quad \sigma_x = (\sigma_x)^{-1}$

It's so simple it's almost confusing! To change

σ_x into I on the left side you just interchange rows!

and σ_x . Doing the same operation to I gives σ_x^{-1}

which is then revealed to be σ_x itself!

2-1

Doing just one of these completely

$$[S_x, S_y] = \frac{\hbar^2}{4} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$
$$\begin{pmatrix} +i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$= \frac{\hbar^2}{4} 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \hbar c \frac{\hbar}{2} \sigma_z = \hbar i S_z.$$

The result for the others:

$$[S_y, S_z] = \hbar i S_x$$

$$[S_z, S_x] = \hbar i S_y$$

3-)

The key observation is $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$

Thus $\sigma_x^3 = \sigma_x \sigma_x^2 = \sigma_x$ etc and

$$\begin{aligned} A &= e^{-it\sigma_x} = I - it\sigma_x + \frac{1}{2}(-it\sigma_x)^2 + \frac{1}{6}(-it\sigma_x)^3 + \dots \\ &= I \left\{ 1 - \frac{1}{2}t^2 + \dots \right\} - i\sigma_x \left\{ t - t^3/6 + \dots \right\} \\ &= I \cos t - i\sigma_x \sin t \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos t - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin t \end{aligned}$$

$$A = \begin{pmatrix} \cos t & -i\sin t \\ -i\sin t & \cos t \end{pmatrix}$$

It turns out to be very important that A is unitary, ie

$$AA^+ = I \text{ with } A^+ = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}$$

B and C are the same calculation. It's pretty

obvious from that

$$B = I \cos t - i\sigma_y \sin t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$$C = I \cos t - i\sigma_z \sin t = \begin{pmatrix} \cos t - i\sin t & 0 \\ 0 & \cos t + i\sin t \end{pmatrix} = \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix}$$

$$\text{again } BB^+ = CC^+ = I$$

Possible values of observable \rightarrow eigenvalues of associated matrix. These are clearly $\pm\hbar, 0$ for S_z .

For S_x :

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 1) - 1(-\lambda - 0)$$

$$= -\lambda^3 + \lambda + \lambda = -\lambda(\lambda^2 - 2)$$

$$\lambda = 0, \pm\sqrt{2}$$

need to multiply by $\hbar/\sqrt{2}$: $\lambda = 0, \pm\hbar$

Same as S_z ! This is expected by symmetry.

Like wise for S_y

$$\begin{vmatrix} -\lambda & -i & 0 \\ i & -\lambda & -i \\ 0 & i & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 1) + i(-i\lambda)$$

$$= -\lambda^3 + \lambda + \lambda$$

$$= -\lambda(\lambda^2 - 2)$$

$\Rightarrow 0, \pm\hbar$ after multiplying by $\hbar/\sqrt{2}$

The eigenvectors of S_x are, for $\lambda = 0$ $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}}$

For $\lambda = t$ (getting rid of $t/\sqrt{2}$) $\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix}$

$$\rightarrow \begin{pmatrix} 1 & -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1/\sqrt{2} & 0 \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

so $v_1 - 1/\sqrt{2} v_2 = 0$

$v_2 - \sqrt{2} v_3 = 0$

and eigenvector is

$$\begin{pmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{pmatrix}$$

notice this is \perp to $\lambda = 0$ eigenvector

as must be the case since S_x is Hermitian

Final eigenvector for $\lambda = -t$

$$\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1/\sqrt{2} & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$v_1 + 1/\sqrt{2} v_2 = 0$

$v_2 + \sqrt{2} v_3 = 0$

$$\begin{pmatrix} -1/\sqrt{2} \\ 1 \\ -1/\sqrt{2} \end{pmatrix}$$

This is \perp to other two eigenvectors v_i

We need to find a_1, a_2, a_3 such that

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = a_1 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + a_2 \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + a_3 \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

\curvearrowleft
invert this matrix or just work with signs for
 a_1, a_2, a_3 directly

$$\frac{1}{\sqrt{2}} a_1 + \frac{1}{2} a_2 - \frac{1}{2} a_3 = \frac{1}{\sqrt{3}} \quad \text{add top + bottom}$$

$$\frac{\sqrt{2}}{2} a_2 + \frac{\sqrt{2}}{2} a_3 = \frac{1}{\sqrt{3}} \quad a_2 - a_3 = \frac{2}{\sqrt{3}}$$

$$-\frac{a_1}{\sqrt{2}} + \frac{1}{2} a_2 - \frac{1}{2} a_3 = \frac{1}{\sqrt{3}}$$

combine with middle

$$\sqrt{2} a_2 = \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} \frac{\sqrt{2}}{2} = \frac{1+\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{3}+\sqrt{6}}{3}$$

$$a_2 = \frac{\sqrt{3}+\sqrt{6}}{3\sqrt{2}} = \frac{\sqrt{6}+\sqrt{12}}{6} = \frac{\sqrt{6}+2\sqrt{3}}{6}$$

$$a_3 = a_2 - \frac{2}{\sqrt{3}} = \frac{\sqrt{6}+\sqrt{12}-4\sqrt{3}}{6} = \frac{\sqrt{6}-2\sqrt{3}}{6}$$

$$a_1 = \frac{\sqrt{2}}{\sqrt{3}} - \frac{\sqrt{2}}{2} a_2 + \frac{\sqrt{2}}{2} a_3 = \frac{1}{12} \{ 4\sqrt{6} - \sqrt{12} - 2\sqrt{6} + \sqrt{12} - 2\sqrt{6} \}$$

$$= \frac{1}{12} \{ 4\sqrt{6} - 2\sqrt{6} \} = \frac{2\sqrt{6}}{12} = \frac{\sqrt{6}}{6}$$

4-4

The probabilities of measuring $S_x = 0, +\hbar, -\hbar$
are $|q_1|^2, |q_2|^2, |q_3|^2$ respectively.

$$|q_1|^2 = p_1 = \frac{0}{3} = .0000 \leftarrow \text{prob}(S_x = 0)$$

$$|q_2|^2 = p_2 = \frac{3+2\sqrt{2}}{6} = .9714 \leftarrow \text{prob}(S_x = +\hbar)$$

$$|q_3|^2 = p_3 = \frac{3-2\sqrt{2}}{6} = .0286 \leftarrow \text{prob}(S_x = -\hbar)$$

These sum to 1
as they must.

In retrospect it's obvious that $q_1 = 0$ because
we could take dot product of ψ eqn on page 2-3

with $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and get

$$\phi = q_1 + \phi + \phi$$

5-1

6-1

$$\boxed{5} \quad S_x^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$S_y^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$S_z^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_0 \quad S^2 = S_x^2 + S_y^2 + S_z^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One can only measure $2\hbar^2$ for S^2 !

$$\boxed{6} \quad S_x^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad S_y^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad S_z^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

One can only measure $\frac{3}{4}\hbar^2$ for S^2 !

7-1

8-1

7

$$\begin{aligned}
 & [S_x, S_y] \\
 &= \frac{\hbar^2}{2} \left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right] \\
 &= \frac{\hbar^2}{2} \left[\begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix} \right] \\
 &= \frac{\hbar^2}{2} 2i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = i\hbar S_z
 \end{aligned}$$

This is the exact same relation as for the spin $\frac{1}{2}$ matrices! Other commutators also are trivial

and one finds $[S_y, S_z] = i\hbar S_x$ $[S_z, S_x] = i\hbar S_y$

8 The general result is $S^{12} = \frac{\hbar^2}{4} S(S+1) I$

e.g. spin $S = \frac{1}{2}$ $S(S+1) = \frac{3}{4}$

appropriately
sized identity
matrix

$$S=1 \quad S(S+1)=2$$