PHYSICS 104A, FALL 2016
MATHEMATICAL PHYSICS

Assignment Four, Due Friday, October 21, 5:00 pm.

[1.] In understanding the behavior of spin-1/2 particles in quantum mechanics, you will encounter the 2×2 'Pauli matrices'

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Compute the inverses of σ_x, σ_y, and σ_z.

[2.] The 'commutator' of two matrices A and B is symbolized by $[A, B]$ and is defined by

$$[A, B] = AB - BA$$

Show that the 'spin' matrices

$$S_x = \frac{\hbar}{2}\sigma_x \qquad S_y = \frac{\hbar}{2}\sigma_y \qquad S_z = \frac{\hbar}{2}\sigma_z$$

obey

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

[3.] to understand how spin wave functions evolve in time t you will need the exponentials of the Pauli matrices,

$$A = e^{-it\sigma_x} \qquad B = e^{-it\sigma_y} \qquad C = e^{-it\sigma_z}$$

Using the definition

$$e^M = I + M + \frac{1}{2}M^2 + \frac{1}{6}M^3 + \cdots$$

compute A, B and C.

[4.] In understanding the behavior of spin-1 particles in quantum mechanics, you will encounter the 3×3 matrices

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

What are the possible values you could get if you measure the x, y or z component of spin of a spin-one particle in an experiment? If your system is in the state $|\psi\rangle = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, what are the probabilities of measuring the different possible values of S_x?

[5.] Construct the matrix $S^2 = S_x^2 + S_y^2 + S_z^2$ for the spin of a spin-one particle. What possible values can you get if you measure the square of the spin?

[6.] Go back to the spin-1/2 matrices of problems 1-3 and similarly construct the matrix $S^2 = S_x^2 + S_y^2 + S_z^2$. What possible values can you get if you measure the square of the spin?

[7.] Compute the commutators $[S_x, S_y]$, $[S_y, S_z]$, and $[S_z, S_x]$ for spin-1. How do the results compare to the spin-1/2 case?

Note: It's a bit weird that when you measure S^2 it is not really what you would naively expect. The reason is that S_x, S_y and S_z do not commute, and so you cannot measure them at the same time. Thus you cannot really hope to get S^2 by just taking the components, squaring them, and adding.

[8.] Extra Credit: You have (above) the matrices S^2 for $S = 1/2$ and for $S = 1$. What is the form of the matrix for the S^2 for a particle of general spin S? This same result will come up in Physics 115AB when you figure out the eigenvalues of the orbital angular momentum of a electron going around a proton in the Hydrogen atom, since the angular momentum matrices obey the same basic rules as spin matrices. You saw it here first.