

PHYSICS 104A, FALL 2018
MATHEMATICAL PHYSICS

Assignment Three, Due Friday, October 19, 5:00 pm.

[1.] Let $\vec{F} = 2xy^2\hat{i} + x^2\hat{j}$. Compute $\int_P \vec{F} \cdot d\vec{r}$ from the origin to $(1, 1)$ (a) along the path $y = x^2$; and (b) along the path $y = x$. Comment on your results. Are they equal? Unequal? What could you have done before even trying to get numerical results to decide whether the two paths would be expected to yield the same answer?

[2.] Find the work done by the force $\vec{F} = x\hat{i} + y\hat{j}$ as one moves along the semicircle of diameter 2 centered at the origin starting from $(1, 0)$ and ending at $(-1, 0)$. How much work is done in returning to the starting point along the diameter? What does this suggest about the force? Verify your conjecture.

[3.] Show that the area enclosed by a counterclockwise curve C in the plane is given by

$$A = \frac{1}{2} \oint_C (x dy - y dx)$$

Verify the formula works for the triangle with vertices $(1, 0)$, $(0, 2)$, $(-1, 0)$.

[4.] Compute

$$\int_0^{2\pi} \frac{d\theta}{13 + 5 \sin \theta}$$

[5.] Compute

$$\int_0^\infty \frac{x^2 dx}{x^4 + 16}$$

This problem closely parallels one of the examples in class. Nevertheless, explain independently the various logical steps you go through, e.g. to get a closed contour to which you can apply the ideas of complex integration. Can you do this problem by a method used in one of your Math 21 classes, such as a trigonometric substitution?

[6.] The “Fermi function” $f(E) = 1/(e^{\beta E} + 1)$ plays a central role in the description of electrons in solids. It gives the number of electrons in a state of energy E if the system has temperature $k_B T = 1/\beta$. (k_B is Boltzmann’s constant.) If E is allowed to be a complex number, locate the poles of $f(E)$.

Extra credit (and very tricky): Evaluate $(1/\beta) \sum_n 1 / (i\omega_n - E)$. Here $\omega_n = \pi(2n+1)/\beta$ are the “Matsubara frequencies” and n are all the integers. This is a very important identity you will learn about in a 2nd year solid state physics grad course. It is very advanced stuff.

1-1

Physics 104A Fall 2018

Problem Set 3

I

Along $y = x$

$$dy = dx$$

$$\int_{(0,0)}^{(1,1)} 2xy^2 dx + x^2 dy$$

$$= \int_0^1 2x^3 dx + x^2 dx = \frac{1}{2}x^4 + \frac{1}{3}x^3 \Big|_0^1 = \frac{5}{6}$$

Along $y = x^2$

$$dy = 2x dx$$

$$= \int_0^1 2x^5 dx + 2x^3 dx = \frac{1}{3}x^6 + \frac{1}{2}x^4 \Big|_0^1 = \frac{5}{6}$$

We get the same answer! But we are surprised

because the force is not conservative!

$$\frac{\partial}{\partial y} F_x = 4xy \quad \frac{\partial}{\partial x} F_y = 2x$$



different.

We speculate we might have just "gotten lucky".

1-2

so we try $(0,0)$ to $(2,4)$ along $y=x$
 $dy=dx$

$$\int_{(0,0)}^{(2,4)} 2xy^2 dx + x^2 dy$$

$$= \int_0^2 2x^3 dx + x^2 dx = \frac{1}{2}x^4 + \frac{1}{3}x^3 \Big|_0^2$$

$$= 8 + 8/3 = 32/3$$

And along $y=x^2$ obviously (from page 1-1)

$$\frac{1}{3}x^6 + \frac{1}{2}x^4 \Big|_0^2 = 64/3 + 8 = 88/3.$$

So it was just an accident that the $(0,0)$ to $(1,1)$

work was the same. This force is not conservative.

also

You can try other paths from $(0,0)$ to $(1,1)$ like $y=x^3$
 \wedge
 $dy=3x^2 dx$

$$\int_0^1 2x^7 dx + 3x^3 dx = \frac{x^8}{4} + \frac{3x^4}{4} \Big|_0^1$$

$$= 1/4 + 3/4 = 1.$$

2-1

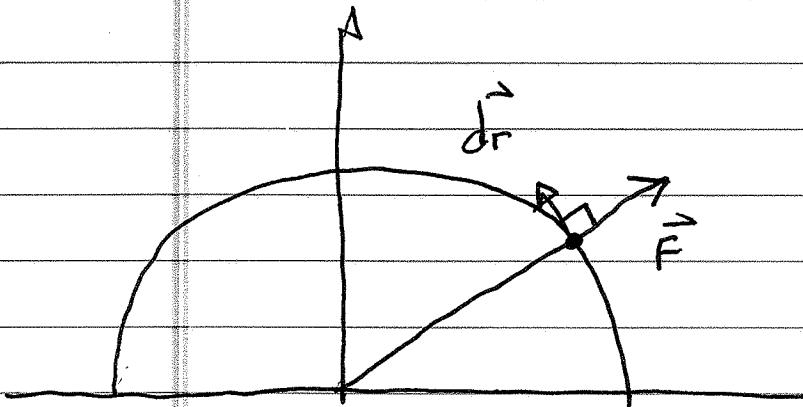
Along the semicircle parameterize the path

$$x = \cos \theta \quad dx = -\sin \theta d\theta \quad 0 < \theta < \pi$$
$$y = \sin \theta \quad dy = \cos \theta d\theta$$

$$\int x dx + y dy = \int_0^\pi \cos \theta (-\sin \theta) d\theta + \sin \theta (\cos \theta d\theta)$$
$$= 0 !$$

In fact, the dW along every little piece of
the path vanishes! This is because $\vec{F} \perp \vec{dr}$

for a circular path and a central force



Along the diameter we expect to get ϕ

because F is conservative $\frac{\partial}{\partial x} F_y = \frac{\partial}{\partial x} y = 0$

and $\frac{\partial}{\partial y} F_x = \frac{\partial}{\partial y} x = 0$.

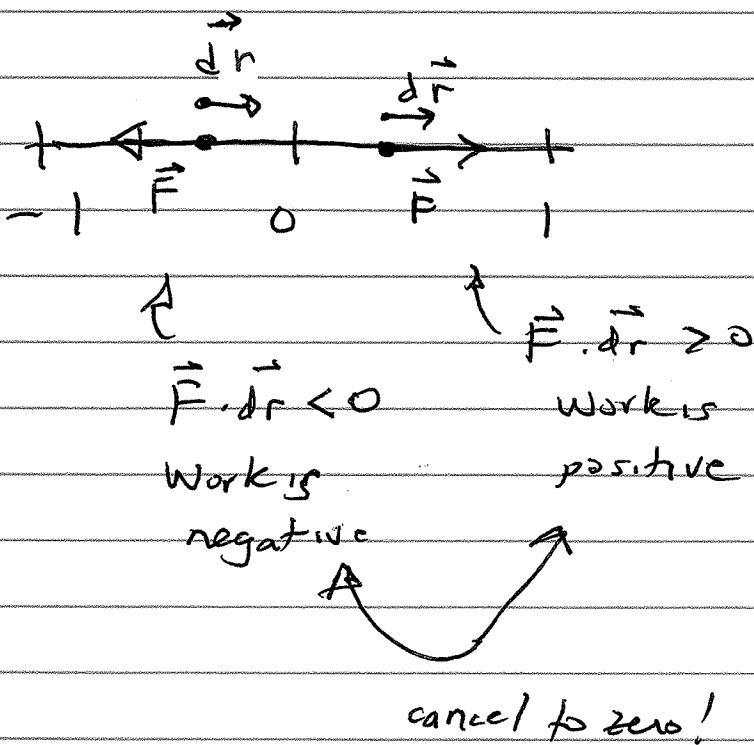
2.2

path

Doing the integral $y = x$ $dy = x$
 $-1 \leq x \leq 1$

$$\int x dx + y dy = \int_{-1}^1 x dx = \frac{1}{2} x^2 \Big|_{-1}^1 = 0$$

Physically:



3-1

By Green's Theorem

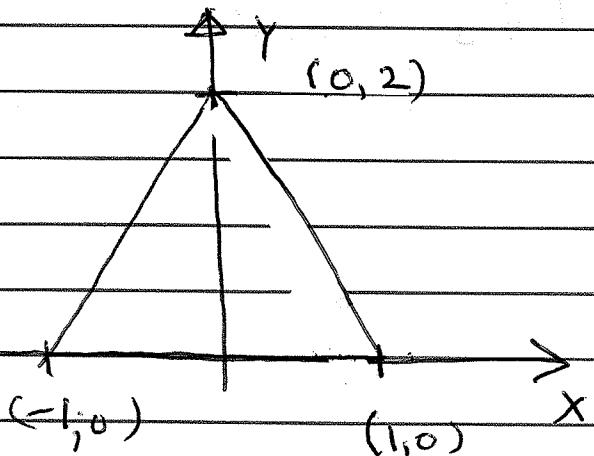
$$\oint_C F_x dx + F_y dy = \iint_A \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx dy$$

Choosing $F_x = -y$ and $F_y = -x$ we have

$$\oint_C x dy - y dx = \iint_A (1 - (-1)) dx dy$$

$$= 2 \iint_A dx dy = 2(\text{Area})$$

$$\therefore \text{Area} = \frac{1}{2} \oint_C x dy - y dx$$



Here C consists of 3 pieces

$$C_1: -1 \xrightarrow{x} 1 \quad y=0$$

$$C_2: \quad y = 2 - 2x \quad 1 \xrightarrow{x} 0$$

$$C_3: \quad y = 2 + 2x \quad 0 \xrightarrow{x} -1$$

The choices make us go around
 C counter-clockwise

$$C_1: \quad \int x dy - y dx = 0 \quad \text{since } dy=0 \text{ and } y=0$$

$$C_2: \quad \int x dy - y dx = \int_1^0 x(-2dx) - (2-2x)dx$$

$\uparrow dy \quad \uparrow y$

4-1

4

We did something very similar in class

$$\text{Write } \sin \theta = (e^{i\theta} - e^{-i\theta})/2i = \frac{1}{2i}(z - 1/z)$$

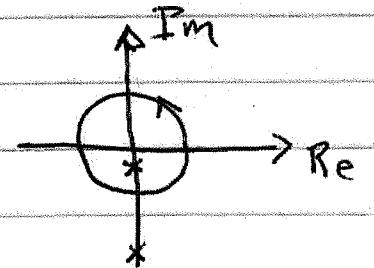
$$\text{Also if } z = e^{i\theta} \quad dz = ie^{i\theta}d\theta = izd\theta$$

so then

$$\oint_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta} = \oint \frac{dz/iz}{13 + \frac{5}{2z}(z - 1/z)}$$

unit circle around
origin

$$= \oint \frac{dz}{\frac{5}{2}z^2 + 13iz - 5/2}$$



$$\left. \begin{array}{l} \text{evaluate eqn} \\ z = \frac{-13i \pm \sqrt{-169 + 25}}{5} \end{array} \right|$$

$$= \oint \frac{dz}{\frac{5}{2}(z + i/5)(z + 5i)}$$

$$= \frac{-13i \pm 120}{5} = \frac{-i}{5}, -5i$$

We will get $2\pi i B_{-1}$ where

$$\frac{1}{\frac{5}{2}(z + i/5)(z + 5i)} = B_{-1} \frac{1}{z + \frac{i}{5}} + B_0 + B_1(z + \frac{i}{5}) + \dots$$

$$\Rightarrow \frac{1}{\frac{5}{2}(z + 5i)} = B_{-1} + B_0(z + \frac{i}{5}) + B_1(z + \frac{i}{5})^2 + \dots$$

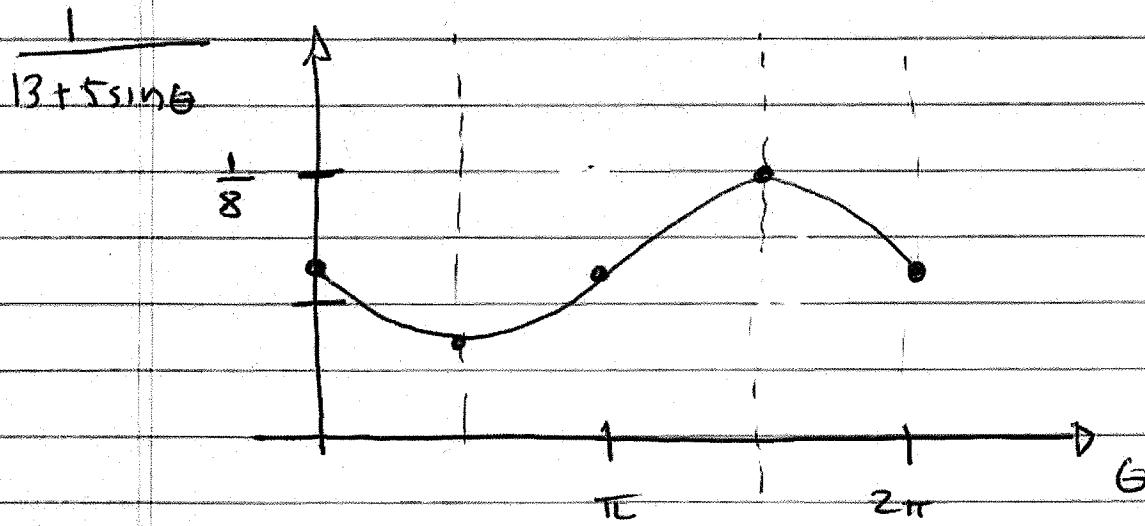
4.2

Setting $z = -i/\zeta$ we see

$$B_{-1} = \frac{1}{\zeta(-\frac{i}{\zeta} + 5i)} = \frac{a}{-i + 25i} = \frac{1}{12i}$$

Thus integral is $2\pi i \frac{1}{12i} = \frac{2\pi}{12} = \pi/6$

As in class we can check with picture



$$\text{Integral} \sim 2\pi \frac{3}{4} \frac{1}{8} \sim \frac{3\pi}{32}$$

close to $\pi/6$.

5-1

5

As in class, use even-ness of integrand

$$\int_0^\infty \frac{x^2}{x^4 + 16} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2}{x^4 + 16} dx$$

Again as in class

$$= \frac{1}{2} \int_C \frac{z^2}{z^4 + 16} dz$$

along real axis where $z = x$

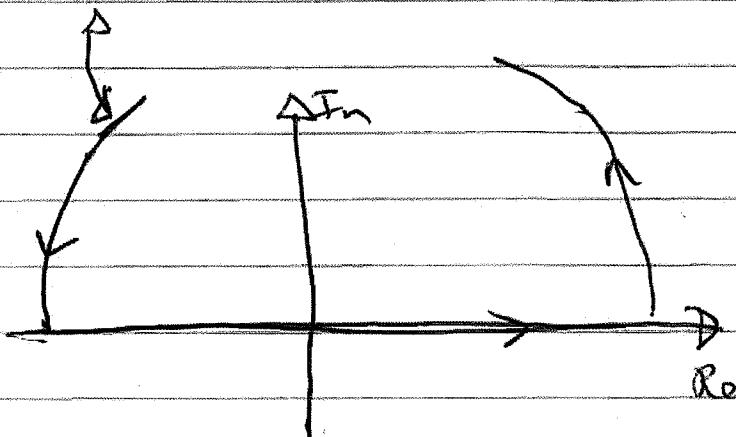
Then add to this an integral along huge semicircle

$z = Re^{i\theta}$ with $R = \infty$ where $z^2/z^4 + 16$ vanishes

(hence you are adding zero!)

End result, the

$$\int_0^\infty \frac{x^2}{x^4 + 16} dx = \frac{1}{2} \oint_C \frac{z^2}{(z^4 + 16)} dz$$



5-2

Next we locate poles (places where

$z^2/(z^4+16)$ is badly behaved) These are

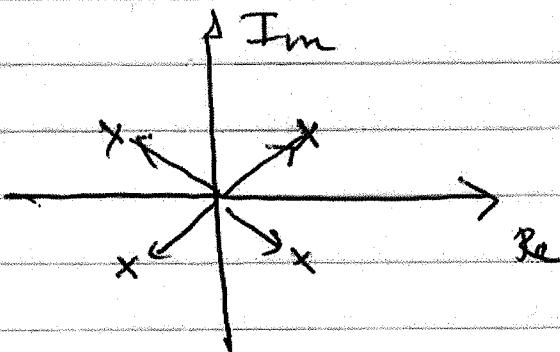
solutions of $z^4 + 16 = 0$

$$z^4 = -16 = +16e^{i\pi}$$

$$z = 2e^{\frac{i\pi}{4}} = \sqrt{2} + \sqrt{2}i$$

but also $-16 = 16e^{3i\pi}, 16e^{5i\pi}, 16e^{7i\pi}$

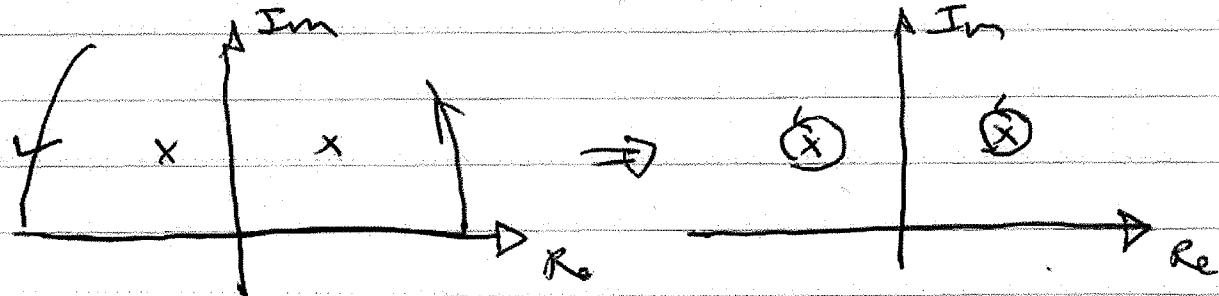
So poles are:



Deforming contour gives us two contributions,

from poles at $z = \sqrt{2} + \sqrt{2}i$ and $-\sqrt{2} + \sqrt{2}i$

We need values of B_+ for each.



5-5

```
#include <stdio.h>
#include <math.h>

int main()
{
    int i,N;
    double integral=0.0,dx,x;

    printf(" \n Enter N,dx ");
    scanf("%i %lf",&N,&dx);

    for (i=0;i<N;i++)
    {
        x=dx*i;
        integral = integral + x*x / ( 1.0 + x*x*x*x );
    }
    integral=integral*dx;
    printf("\n integral is %12.8lf \n",integral);
}
```

The physics 102 Method:

Results (Need to check convergence
 both with Δx small and also going out
 $\rightarrow x_{max} = \infty$)

N	Δx	$x_{max} = N \Delta x$	integral
100	0.1	10	1.01022
200	0.1	20	1.06060
500	0.1	50	1.09070
1000	0.1	100	1.10072
2000	0.1	200	1.10572
4000	0.05	200	1.10572
10^6	10^{-3}	1000	1.10972
10^6	$5 \cdot 10^{-3}$	5000	1.11052
			1.1107207 $\leftarrow \frac{\pi}{2\sqrt{2}}$

notice we

need much larger N here

than problem 6. There are tricks for dealing with that,

Trick for using smaller N :

Can compute "tail" at large x

$$\int_{x_{\max}}^{\infty} \frac{x^2 dx}{1+x^4} = \int_{x_{\max}}^{\infty} \frac{x^2}{x^4(1+\frac{1}{x^4})} dx$$

$$= \int_{x_{\max}}^{\infty} \frac{1}{x^2} \left(1 - \frac{1}{x^4} + \frac{1}{x^8} - \dots\right) dx$$

$$= -\frac{1}{x} + \frac{1}{5x^5} - \dots \Big|_{x_{\max}}^{\infty}$$

$$= \frac{1}{x_{\max}}$$

So integrate out to x_{\max} and add

analytic correction.

5-7

```
#include <stdio.h>
#include <math.h>

int main()
{
    int i,N;
    double integral=0.0,dx,x;

    printf(" \n Enter N,dx ");
    scanf("%i %lf",&N,&dx);

    for (i=0;i<N;i++)
    {
        x=dx*i;
        integral = integral + x*x / ( 1.0 + x*x*x*x );
    }
    integral=integral*dx;
    printf("\n integral is %12.8lf \n",integral);
    integral=integral+1.0/(N*dx);
    printf("\n integral plus correction is %12.8lf \n",integral);
}
```

N	dx	Δx	Integral	$Integral + \frac{1}{N}x_{max}$
4000	.05	200	1.10572	1.11072

$$\text{exact} = 1.11072$$

b-1

[b] The denominator $e^{\beta E} + 1 = 0$ when $e^{\beta E} = -1$

This occurs at energies $E = i\pi/\beta(2n+1)$ for $n = \text{integer}$

As stated in the extra credit description, these are

called "MatSUBARA frequencies"

Extra credit The 1st key observation is that

the Fermi function $f(E) = 1/(e^{\beta E} + 1)$ has poles

at the MatSUBARA frequencies (as we just saw). In fact,

the residues are obtained by

$$e^{\beta E} + 1 = e^{\beta(E-i\omega_n)+i\omega_n} + 1 = e^{i\omega_n} e^{\beta(E-i\omega_n)} + 1$$

$\underset{-1}{\cancel{}}$

$$= -e^{\beta(E-i\omega_n)} + 1 = -(1 + \beta(E-i\omega_n) + \dots) + 1$$

$$= -\beta(E-i\omega_n)$$

$$\text{so } (E-i\omega_n) \frac{1}{e^{\beta E} + 1} = -\frac{1}{\beta}.$$

6-2

The 2nd key observation is that if we have

a function $f(z)$ with pole at z_0 and residue B_{-1}

then the function $h(z)f(z)$ has Residue $h(z_0)B_{-1}$

as long as $h(z)$ is well behaved at z_0 . This

follows from our rule for computing residues

$$B_{-1} = (z - z_0) f(z) \Big|_{z=z_0}$$

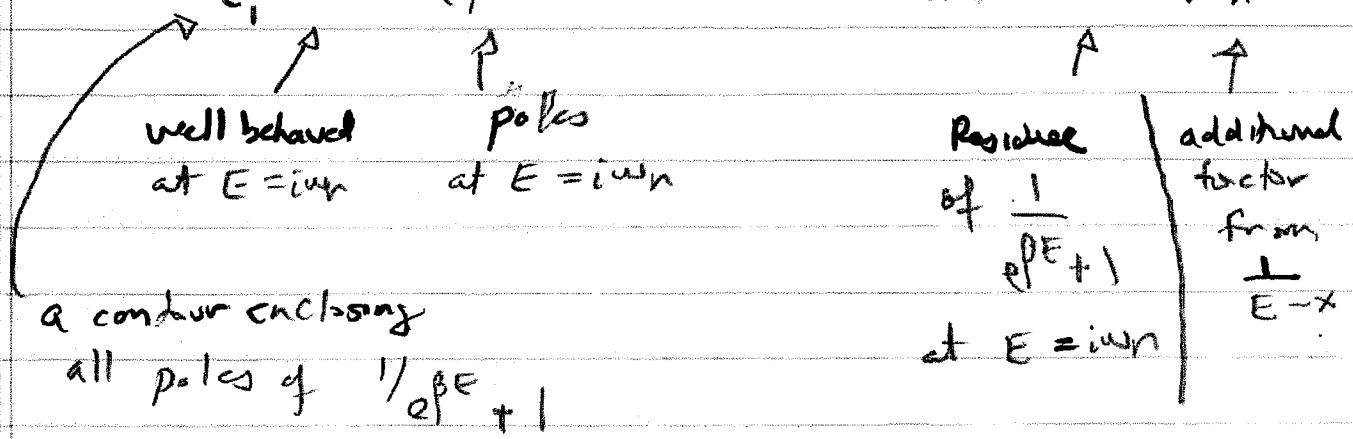
$$\Rightarrow (z - z_0) h(z) f(z) \Big|_{z=z_0} = B_{-1} h(z_0)$$



gives extra factor

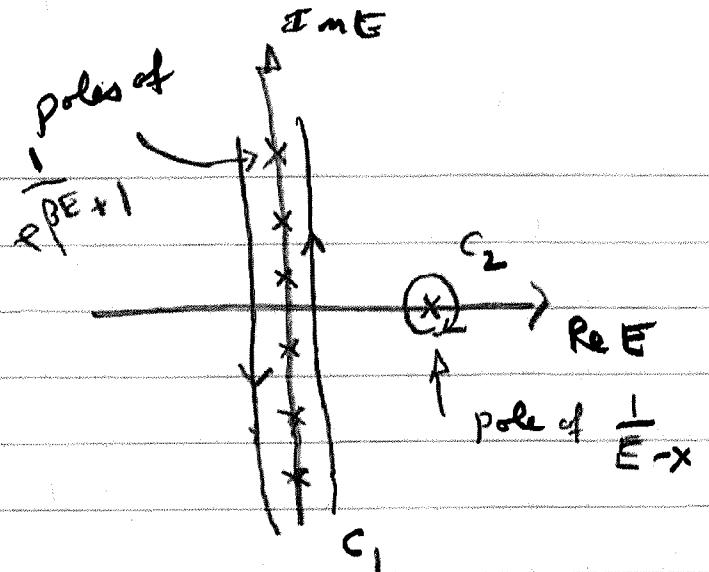
Putting these together we see that

$$\oint_{C_1} \frac{1}{E-x} \frac{1}{e^{\beta E} + 1} dE = 2\pi i \sum_n \left(-\frac{1}{\beta} \right) \frac{1}{i\omega_n - x}$$



6-3

A suitable contour C_1
enclosing all poles of
 $\frac{1}{e^{\beta E} + 1}$ is



From page 3-2

$$\oint_{C_1} \frac{1}{E-x} \frac{1}{e^{\beta E} + 1} dE = -\frac{2\pi i}{\beta} \sum n_i w_n - x$$

Finally we can distort C_1 to C_2 without changing the integral;
since in going from C_1 to C_2 we only pass through
analytic regions.

$$\oint_{C_1} \frac{1}{E-x} \frac{1}{e^{\beta E} + 1} dE \approx \oint_{C_2} \frac{1}{E-x} \frac{1}{e^{\beta E} + 1} dE$$

Now this is the relevant
pole; its residue

is obviously 1 and

$\frac{1}{e^{\beta E} + 1}$ is the "extra
factor"

$$\oint_{C_2} \frac{1}{E-x} \frac{1}{e^{\beta E} + 1} = \underbrace{\frac{1}{e^{\beta x} + 1}}_{\text{Residue}} 2\pi i (-1)$$

↑
clockwise
traversal of C_2

6-4

So now we see

$$-\frac{2\pi i}{\beta} \sum_n \frac{1}{i\omega_n - x} = \frac{1}{e^{\beta x} + 1} 2\pi i (-1)$$
$$\Rightarrow \boxed{\frac{1}{\beta} \sum_n \frac{1}{i\omega_n - x} = \frac{1}{e^{\beta x} + 1}}$$

As noted in the problem set, this is a very fundamental result of the solution of interacting electron problems in a solid by "Feynman diagram methods"