

**PHYSICS 104A, FALL 2018**  
**MATHEMATICAL PHYSICS**

**Assignment Two, Due Friday, October 12, 5:00 pm.**

[1.] Compute  $\cos(i\pi)$ .

[2.] A special case of problem 5 of assignment 1 is that multiplication by  $i$  rotates a complex number by  $90^\circ$  without changing its length. Use this idea in the following problem: Let  $z = Re^{i\omega t}$  describe the displacement from the origin of a particle at time  $t$ . Show that the particle travels in a circle of radius  $R$  at velocity  $v = R\omega$  and with acceleration of magnitude  $v^2/R$  directed towards the center of the circle. Bonus: Find the grammar mistake in this problem.

[3.] For functions of a real variable  $df(x)/dx$ , for  $f(x) = 1/x$ , is  $-1/x^2$ . Compute  $df(z)/dz$ , for  $f(z) = 1/z$ . Is it equal to  $-1/z^2$ ? Are the Cauchy-Riemann conditions satisfied?

[4.] For functions of a real variable  $df(x)/dx$ , for  $f(x) = \sqrt{x}$ , is  $1/(2\sqrt{x})$ . Compute  $df(z)/dz$ , for  $f(z) = \sqrt{z}$ . Is it equal to  $1/(2\sqrt{z})$ ? Are the Cauchy-Riemann conditions satisfied?

[5.] Is the function  $f(z) = z^*$  analytic? (That is, does it satisfy the Cauchy-Riemann conditions?)

[6.] In class we computed the work done by the force  $\vec{F} = 4x\hat{i} - 2y\hat{j}$  in moving a particle from  $(0, 0)$  to  $(1, 2)$  along the two paths  $y = 2x$  and  $y = 2x^2$ . We got the same answer. Using the same force, compute the work in going from  $(-1, 0)$  to  $(2, 1)$

- (a) Along the path  $y = (x + 1)/3$ ;
- (b) Along the path  $y = x^2 - 2x/3 - 5/3$ .

First show that both paths contain the desired endpoints!

[7.] You are told the  $x$  component of a force  $\vec{F}$  is  $F_x = 3xy$ . You are also told that  $\vec{F}$  is conservative, i.e. the work done by  $\vec{F}$  in moving between two points in the  $xy$  plane does not depend on the path. What can you say about  $F_y$ ?

[8.] The more complete analysis of forces and their path dependence uses *Green's Theorem*,

$$\oint F_x dx + F_y dy = \iint (\partial F_x / \partial y - \partial F_y / \partial x) dxdy$$

Here the integral on the left is around a *closed loop* in the  $xy$  plane and the integral on the right is over the *area enclosed by the loop*. Why is this equivalent to the statement from class that if  $\partial F_x / \partial y = \partial F_y / \partial x$  the integral between two points is path independent?

This equation contains a lot of fancy looking symbols, but the problem you are asked to do is surprisingly simple. Without doing any double integrals, you should be able to say something very quickly about the right hand side if  $\partial F_x / \partial y = \partial F_y / \partial x$ . Then you need to figure out what an integral around a closed loop has to do with integrals along different paths between two points. Think about what happens to integrals when you change the direction with which you travel from beginning point to ending point.

## Physics 104A Fall 2016

## Assignment 2

Various ways to do this. The simplest (trivial) way is

$$\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\text{so } \cos(i\pi) = \frac{1}{2}[e^{i(i\pi)} + e^{-i(i\pi)}]$$

$$= \frac{1}{2}(e^{\pi} + e^{-\pi}) = \cosh \pi$$

You could also proceed more from "first principles" i.e. the Taylor expansion for cosine. This lets us realize  $\cos(i\pi)$  is real because only even terms are present

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

$$\cos(i\pi) = 1 + \frac{1}{2!}\pi^2 + \frac{1}{4!}\pi^4 + \frac{1}{6!}\pi^6 + \dots$$

Then you need to realize this is the Taylor expansion for  $\cosh$ .

In general "trig functions" like sine, cosine, tangent are related to hyperbolic functions  $\cosh, \sinh, \tanh$  by taking imaginary arguments.

2-1

The point of this problem is an alternate view of uniform circular motion enabled by complex numbers.

First, note  $z = ae^{i\omega t}$  means

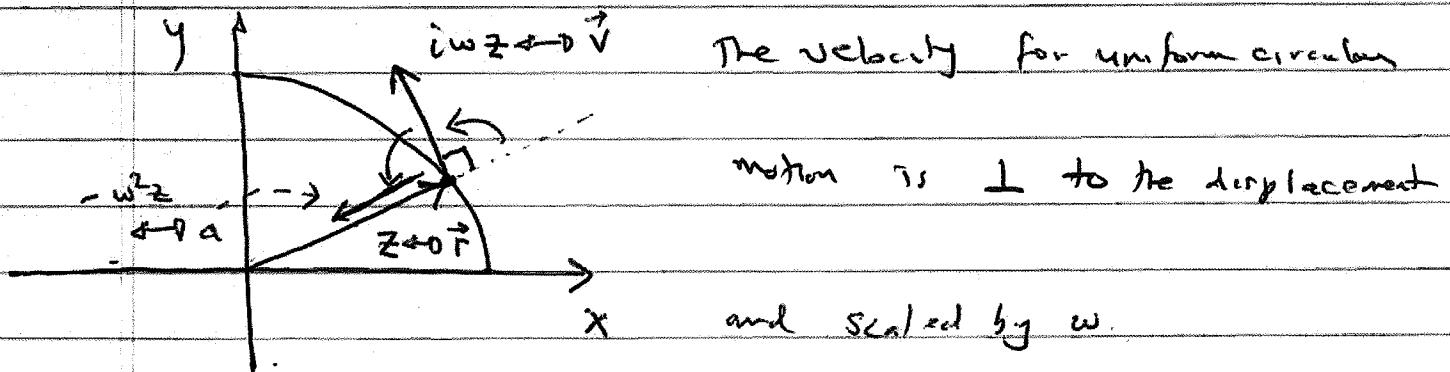
$$x = a \cos \omega t \quad y = a \sin \omega t$$

which gives circular motion.

Now  $\frac{dz}{dt} = i\omega a e^{i\omega t}$  which looks like our  $pe^{i\phi}$  HW#1 problems with  $p = \omega$  and  $\phi = \overline{\theta}_2$

so by that problem  $\frac{dz}{dt}$  is rotated by  $\overline{\theta}_2$  and

scaled by a factor  $\omega$ . We know this is right:



Similarly, differentiating again rotates by another  $\overline{\theta}_2$  and gets scaled by  $\omega$  again, yielding the familiar result that the acceleration for uniform circular motion is directed towards the center of the circle and is  $\omega^2 a$  times the radius.

3-1

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$\text{so } u(x,y) = \frac{x}{x^2+y^2} \quad v(x,y) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} - \frac{x}{(x^2+y^2)^2}(2x)$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = -\frac{1}{(x^2+y^2)} + \frac{y}{(x^2+y^2)^2}(2y)$$

$$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

Clearly  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = -\frac{x}{(x^2+y^2)^2}(2y) = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{y}{(x^2+y^2)^2}(2x) = \frac{+2xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \checkmark$$

so we can evaluate  $\frac{df}{dz}$  with  $dz = dx$   
and be assured the answer is the same for any dz

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} + i \frac{2xy}{(x^2+y^2)}$$

Now check  $\frac{1}{z^2} = \frac{1}{(x+iy)^2} = \frac{1}{x^2-y^2+2ixy}$

3-2

Rationalizing the denominator

$$-\frac{1}{z^2} = \frac{-[(x^2 - y^2) - 2ixy]}{(x^2 - y^2)^2 + (2xy)^2} = \frac{y^2 - x^2 + 2ixy}{(x^2 + y^2)^2}$$

so we see  $\frac{df}{dz} = -\frac{1}{z^2}$  for  $f(z) = \frac{1}{z}$  !

4-1

$$f(z) = z^{1/2} = (re^{i\theta})^{1/2} = r^{1/2} e^{i\theta/2}$$

$$= r^{1/2} \{ \cos \theta/2 + i \sin \theta/2 \}$$

$$u = r^{1/2} \cos \theta/2 \quad v = r^{1/2} \sin \theta/2$$

The algebra is simplest using a mix of cartesian

and polar forms of  $z$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (x^2 + y^2)^{1/2} = \frac{1}{2} 2x(x^2 + y^2)^{-1/2} = x/r$$

$$\frac{\partial v}{\partial y} = y/r \quad \text{similarly}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x} (\tan^{-1} y/x) = \frac{1}{1+y^2/x^2} \left( -\frac{y}{x^2} \right) = -y/r^2$$

$$\frac{\partial \theta}{\partial y} = +x/r^2 \quad \text{similarly}$$

thus  $\frac{\partial u}{\partial x} = \frac{1}{2} r^{-1/2} \frac{x}{r} \cos \theta/2 + r^{1/2} \sin \theta/2 \frac{1}{2} (-y/r^2)$

$$= \frac{1}{2} r^{-3/2} \{ x \cos \theta/2 + y \sin \theta/2 \}$$

$$\frac{\partial v}{\partial y} = \frac{1}{2} r^{-1/2} \frac{y}{r} \sin \theta/2 + r^{1/2} \cos \theta/2 \frac{1}{2} x/r^2$$

$$= \frac{1}{2} r^{-3/2} \{ y \sin \theta/2 + x \cos \theta/2 \}$$

i.  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

4-2

$$\frac{\partial u}{\partial y} = \frac{1}{2} r^{-\frac{1}{2}} y/r \cos \frac{\theta}{2} - r^{\frac{1}{2}} \sin \frac{\theta}{2} \frac{1}{2} x/r^2$$

$$= \frac{1}{2} r^{-\frac{3}{2}} \left\{ y \cos \frac{\theta}{2} - x \sin \frac{\theta}{2} \right\}$$

$$\frac{\partial v}{\partial x} = \frac{1}{2} r^{-\frac{1}{2}} \frac{x}{r} \sin \frac{\theta}{2} + r^{\frac{1}{2}} \cos \frac{\theta}{2} \frac{1}{2} (-y/r^2)$$

$$= \frac{1}{2} r^{-\frac{3}{2}} \left\{ x \sin \frac{\theta}{2} - y \cos \frac{\theta}{2} \right\}$$

$$\therefore \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow$  Cauchy Riemann is satisfied!

It remains to verify whether  $\frac{d}{dz} z^{\frac{1}{2}} = \frac{1}{2} z^{-\frac{1}{2}}$

Let's do  $\partial z = dx$  because CR tells us we get same

answer either way

$$\frac{d}{dz} z^{\frac{1}{2}} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$= \frac{1}{2} r^{-\frac{3}{2}} \left\{ x \cos \frac{\theta}{2} + y \sin \frac{\theta}{2} \right\} + i \frac{1}{2} r^{-\frac{3}{2}} \left\{ x \sin \frac{\theta}{2} - y \cos \frac{\theta}{2} \right\}$$

$$= \frac{1}{2} r^{-\frac{3}{2}} x \left\{ \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right\} + \frac{1}{2} r^{-\frac{3}{2}} y \left\{ \sin \frac{\theta}{2} - i \cos \frac{\theta}{2} \right\}$$

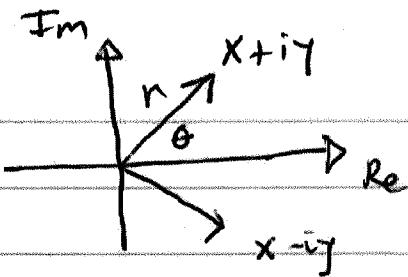
$$= \frac{1}{2} r^{-\frac{3}{2}} x e^{i\theta/2} + \frac{1}{2} r^{-\frac{3}{2}} y (-i e^{-i\theta/2})$$

$$= \frac{1}{2} r^{-\frac{3}{2}} e^{i\theta/2} (x - iy)$$

4-3

But if  $x+iy = re^{i\theta}$

$$x-iy = re^{-i\theta}$$



$$\frac{d}{dz} z^{1/2} = \frac{1}{2} r^{-3/2} e^{i\theta/2} e^{-i\theta}$$

$$= \frac{1}{2} r^{-3/2} e^{-i\theta/2}$$

we began whole calculation with  $z^{1/2} = r^{1/2} e^{i\theta/2}$

$$\frac{d}{dz} z^{1/2} = \frac{1}{2} z^{-1/2} \quad \checkmark$$

WARNING For all this one does need to be careful with  $z^{1/2}$

because there is an ambiguity when you take a

square root of complex #'s, analogous to the

$$\text{ambiguity } \sqrt{4} = \pm 2 \text{ for real #'s}$$

↑  
two answers.

(but complicated!)

We will not go into this interesting issue,

5-1

$$f(z) = z^* = x - iy$$

$$u(x, y) = x$$

$$v(x, y) = -y$$

$$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = -1$$

so Cauchy-Riemann do not hold!

6-1

$$\vec{F} = 4x \hat{i} - 2y \hat{j} \quad \begin{cases} F_x = 4x \\ F_y = -2y \end{cases}$$

Path 1

$$y = \frac{1}{3}(x+1) \quad dy = \frac{1}{3}dx$$

$$\int F_x dx + F_y dy = \int 4x dx - \frac{2}{3}(x+1) \frac{1}{3} dx$$

$$= \int_{-1}^2 \left(4x - \frac{2}{9}x - \frac{2}{9}\right) dx$$

$$= \left(2x^2 - \frac{1}{9}x^2 - \frac{2}{9}x\right) \Big|_1^2$$

$$= \frac{1}{9} \left\{ 72 - 4 - 4 - (18 - 1 + 2) \right\} = \frac{45}{9}$$

Path 2

$$y = x^2 - \frac{2x}{3} - \frac{5}{3} \quad dy = (2x - \frac{2}{3}) dx$$

$$\int F_x dx + F_y dy = \int 4x dx - 2(x^2 - \frac{2x}{3} - \frac{5}{3})(2x - \frac{2}{3}) dx$$

$$= \int (4x - 4x^3 + \frac{4}{3}x^2$$

$$+ \frac{8}{3}x^2 - \frac{8}{9}x$$

$$+ \frac{20}{3}x - \frac{20}{9}) dx$$

$$= \int_{-1}^2 \left(-4x^3 + 4x^2 + \frac{88}{9}x - \frac{20}{9}\right) dx$$

$$= \left(-x^4 + \frac{4}{3}x^3 + \frac{44}{9}x^2 - \frac{20}{9}x\right) \Big|_1^2$$

$$= \frac{1}{9} \left\{ (-144 + 96 + 176 - 40) - (-9 - 12 + 44 + 20) \right\}$$

$$+ 88$$

$$= 43$$

$$= \frac{45}{9} \quad \text{Same answer (and worth the work!)}$$

7-1

A conservative force obeys  $\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$

Here we are given  $F_x = 3xy$  so  $\frac{\partial F_x}{\partial y} = 3x$

Therefore  $\frac{\partial F_y}{\partial x} = 3x$

$$F_y = \frac{3}{2}x^2 + g(y)$$

$\uparrow$  an arbitrary function  
of  $y$ ! This is the  
analogy of the "constant"  
you include in indefinite  
integrals of a single variable

8-1

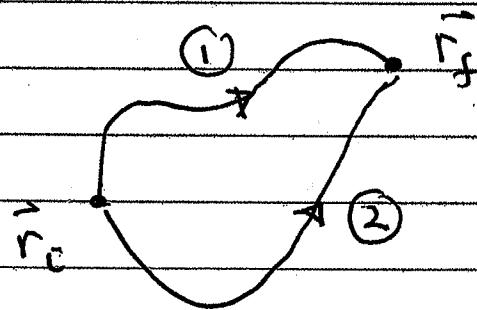
If  $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$  the right hand side must vanish

(\*)

because we are integrating a quantity which is zero!

Now consider integrating between  $\vec{r}_i$  and  $\vec{r}_f$  along

two different paths



The key observation is that if we reverse the way we traverse path ② the integral changes sign.

$$\oint \vec{F} \cdot d\vec{r} = \int_1 \vec{F} \cdot d\vec{r} - \int_2 \vec{F} \cdot d\vec{r} = 0 \quad \text{by (*)}$$

$$\Rightarrow \int_1 \vec{F} \cdot d\vec{r} = \int_2 \vec{F} \cdot d\vec{r} \Rightarrow \text{work is path independent}$$