

PHYSICS 104A, FALL 2018
MATHEMATICAL PHYSICS

Assignment One, Due Friday, October 5, 5:00 pm.

[1.] You can decompose a real number into the sum of two other real numbers in many possible ways: $7=2+5$ but also $7=4+3$, for example. Prove that the decomposition of a complex number into real and imaginary parts is unique.

[2.] We solved the damped harmonic oscillator in class. Do the same to find the charge $Q(t)$ on the capacitor plate in an LRC circuit. How do you incorporate the initial conditions $Q(t = 0)$ and $I(t = 0)$ into your general solution?

[3.] Suppose you had an anharmonic spring $F = -kx - ax^3$. What happens if you try a solution $x(t) = e^{i\omega t}$ in the equation of motion? Is there any analytic method which will solve this problem?

[4.] Compute $(0.6 + 0.8i)^{90}$.

[5.] Consider a complex number z as a vector. Describe in words what happens to z if you multiply it by $\rho e^{i\phi}$. (Here ρ and ϕ are real numbers.) How does the length of $\rho e^{i\phi} z$ compare to that of z ? What about its direction?

[6.] The ground state energy of a one dimensional metallic chain with N atoms is given by

$$E_0 = \sum_{j=-N/4}^{N/4} -2t \cos \frac{2\pi j}{N}$$

Here the parameter t is the ‘hopping integral’ and is determined by the overlap of wavefunctions on adjacent atoms. Depending on the material, t might have a value of, for example, $t = 2.5$ eV for polyacetylene [1]. Find a closed-form expression for this sum. Can you discover what it converges to in the “thermodynamic limit” $N \rightarrow \infty$?

[7.] In class we discussed the solutions of $z^4 = 1$ being $z = 1, i, -1, -i$. Find (a) the solutions to $z^4 = 1 + i$; and (b) the solutions to $z^4 - (3 + i)z^2 + 2 + i = 0$.

[8.] Compute $\ln(3 + 4i)$.

[1] The classic paper on polyacetylene, pointing out the existence of solitons (!), is here: <http://journals.aps.org/prl/pdf/10.1103/PhysRevLett.42.1698>. We will derive the key results of this paper when we go through our discussion of matrices and eigenvalues.

P104A Homework #1

If $a+ib = c+id$ $\leftarrow a, b, c, d \text{ are}$
 $\text{real } \underline{\#}'s$
 then $a-c = i(d-b)$

$$\text{Squaring } (a-c)^2 = -(d-b)^2$$

Since a, b, c, d are real $\rightarrow (a-c)^2$

and $(d-b)^2$ must be ≥ 0 . So the minus

sign tells us (nonnegative) $=$ -(nonnegative)

\Rightarrow the numbers must be zero and $a=c$; $b=d$

An alternate argument is to view $a+ib$ and

$c+id$ as vectors with "x" components a and c

and "y" components b and d . Two vectors can

only be equal if their components match. So, again

$a=c$ and $b=d$.

2-1

$$L \frac{dI}{dt} + IR + \frac{Q}{C} = 0 \quad \begin{matrix} \text{"Kirchhoff's} \\ \text{Law"} \end{matrix}$$

$$I = \overset{\circ}{Q} \quad \begin{matrix} \sum \text{Voltage} = 0 \\ \text{closed} \\ \text{loop} \end{matrix}$$

$$L \overset{\circ}{Q} + R \overset{\circ}{Q} + \frac{1}{C} \overset{\circ}{Q} = 0$$

$$\overset{\circ}{Q} = Q_0 e^{i\omega t}$$

$$Q_0 e^{i\omega t} \left\{ -\omega^2 L + iR\omega + \frac{1}{C} \right\} = 0$$

$$\Rightarrow \omega = \frac{-iR \pm \sqrt{-R^2 + 4/L}}{-2L}$$

$$\omega = \frac{iR}{2L} \pm \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$



time constant

for charge amplitude

$$\text{decay } \tau = R/2L$$

frequency is

$$\sqrt{1/LC}$$

 R reduces this

2-2

It is probably most simple to rederive do sine, cosine form to incorporate initial conditions. Put better, Real & Imaginary parts of our guessed soln $x_0 e^{i\omega t}$ are both solns

$$Q(t) = (A \cos \omega t + B \sin \omega t) e^{-Rt/2L}$$

Clearly $Q(t=0) = Q_0 = A$

and since

$$I(t) = (-\omega A \sin \omega t + \omega B \cos \omega t) e^{-Rt/2L} + (A \cos \omega t + B \sin \omega t)(e^{-Rt/2L})(-R/2L)$$

$$I(t=0) = B\omega + A(-R/2L)$$

$$I_0 = B\omega - Q_0 R/2L$$

$$B = (I_0 + Q_0 R/2L) / \omega$$

3-1

$$\ddot{mx} + kx + \alpha x^3 + \gamma \dot{x} = 0$$

guess $x(t) = x_0 e^{i\omega t}$

$$x_0 \left\{ -m\omega^2 e^{i\omega t} + ke^{i\omega t} + \alpha x_0^2 e^{3i\omega t} + i\gamma \omega e^{i\omega t} \right\} = 0$$

↑
This $e^{3i\omega t}$ spoils
things completely

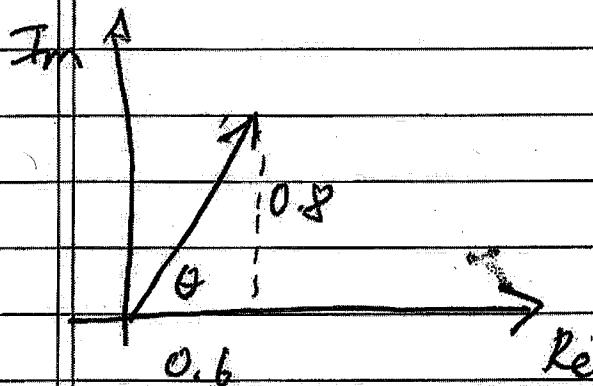
General message is very important: "Nonlinear"

differential eqns are much harder (impossible?)

to solve than linear ones.

4-1

$$(0.6+0.8i) = e^{i\theta} \text{ with } \tan\theta = \frac{4}{3}$$



$$\therefore (0.6+0.8)^{90} = e^{i90\theta}$$

$$\theta = .927295 \text{ radians}$$

$$90\theta = 83.456567$$

$$\mod 2\pi$$

$$= 1.775161$$

$$(0.6+0.8)^{90} = e^{i(1.775161)}$$

$$= -0.202945 + 0.979190 i$$

5-1

If we also write z in polar form $z = r e^{i\theta}$

then $p e^{i\phi} z = p e^{i\phi} r e^{i\theta} = p r e^{i(\phi+\theta)}$

Thus z is changed in length by the factor p

and its direction is changed by angle ϕ .

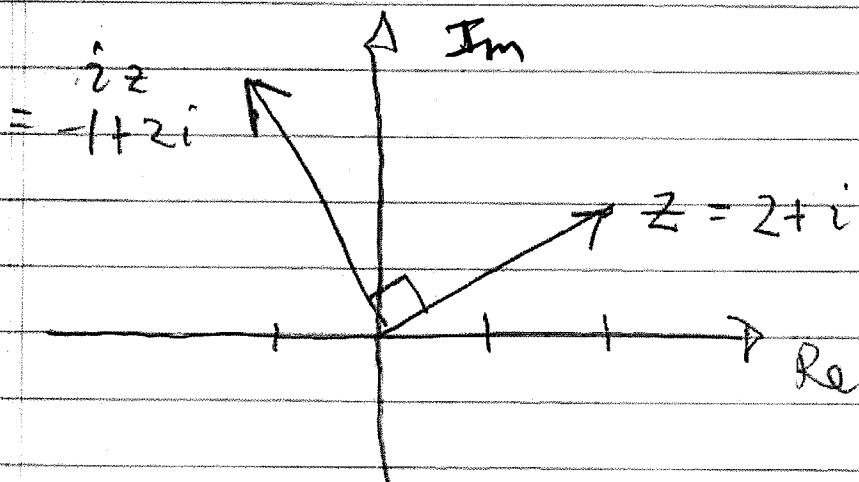
Multiplication by i is an interesting special

case $i = 1 e^{i\pi/2}$ so $p=1$ and $\phi = \pi/2$

Multiplication by i leaves length of z unaltered

and rotates z by $\pi/2$ (90 degrees)

$$z = 2+i \quad iz = -1+2i$$



5-1

$$E_0 = \sum_{j=-N/4}^{N/4} -2t \cos \frac{2\pi j}{N}$$

NB: This same
sort of sum
arises in diffraction
problems !

$$= -2t \operatorname{Re} \sum_{j=-N/4}^{N/4} e^{i 2\pi j / N}$$

$$= -2t \operatorname{Re} e^{-i \pi / 2} \sum_{j=0}^{N/2} e^{i 2\pi j / N}$$

$$= -2t \operatorname{Re} \left\{ -i (1 + x + x^2 + \dots + x^{N/2}) \right\}$$

with $x = e^{i 2\pi / N}$

usual
tech

$$S = 1 + x + x^2 + \dots + x^{N/2}$$

$$xS = x + x^2 + \dots + x^{N/2+1}$$

$$(1-x)S = 1 - x^{N/2+1}$$

$$S = \frac{1 - x^{N/2+1}}{1 - x} = \frac{1 - e^{i 2\pi / N (N/2+1)}}{1 - e^{i 2\pi / N}}$$

$$= \frac{(1 - e^{i 2\pi / N (N/2+1)}) (1 - e^{-i 2\pi / N})}{(1 - e^{i 2\pi / N}) (1 - e^{-i 2\pi / N})}$$

6-2

$$S = \frac{\underbrace{1 + e^{i\pi} - e^{i\pi + i2\pi/N} - e^{-i2\pi/N}}_{2 - 2\cos^2\pi/N}}{1 + 1 - e^{i2\pi/N} - e^{-i2\pi/N}}$$

$$= \frac{e^{i2\pi/N} - e^{-i2\pi/N}}{2(1 - \cos^2\pi/N)} \quad \leftarrow 2i\sin^2\pi/N$$

$$S = i \frac{\sin^2\pi/N}{1 - \cos^2\pi/N}$$

$$E_0 = -2t \operatorname{Re} \{ -i \}$$

$$= -2t \operatorname{Re} \left\{ \frac{\sin^2\pi/N}{1 - \cos^2\pi/N} \right\} = -2t \frac{\sin^2\pi/N}{1 - \cos^2\pi/N}$$

In the limit $N \rightarrow \infty \quad \frac{2\pi}{N} \rightarrow 0$

$$\sin x \approx x$$

$$1 - \cos x \approx 1 - \left(1 - \frac{x^2}{2}\right) = \frac{x^2}{2}$$

$$E_0 = -2t \frac{\frac{2\pi}{N}}{\frac{1}{2}(2\pi)^2/N^2} = -\frac{2}{\pi} t N$$

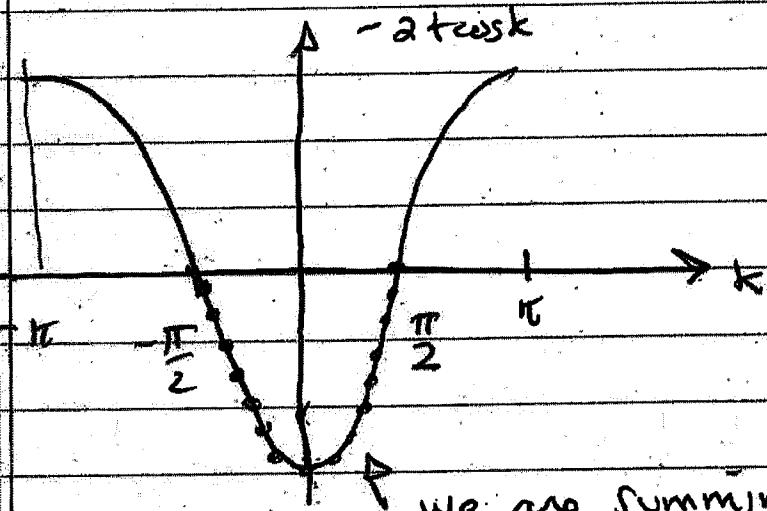
$$E_0 \sim N$$

the # of atoms $\propto L^1$

6-3

 $N \rightarrow \infty$

Can check this result another way



We are summing $-2t \cos k$
on an equally spaced grid of points

$$-\frac{\pi}{2} < k \leq \frac{\pi}{2} \text{ spacing } \frac{2\pi}{N}$$

This is just the Riemann sum for $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -2t \cos k dk$

That is

$$\frac{2\pi}{N} \sum_{j=-N/4}^{N/4} -2t \cos \frac{2\pi j}{N} = \frac{2\pi}{N} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -2t \cos k dk$$

(dk)

$$= -2t \sin k \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4t$$

$$E_0 = \frac{N}{2\pi} (-4t) = -\frac{2t}{\pi} N$$

7-1

[6]

$$z^4 = 1+i = \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right)$$

$$= \sqrt{2} e^{i\pi/4} = 2^{1/2} e^{i\pi/4}$$

↑

Can add $2\pi, 4\pi, 6\pi$ to angle $\pi/4$ without changing exponential since $e^{i\theta} = \cos \theta + i \sin \theta$ and \cos, \sin have period 2π .

$$z = 2^{1/8} e^{i\pi/16} \{ 1, e^{i\pi/2}, e^{i\pi}, e^{i3\pi/2} \}$$

↑ ↑ ↑

additional $2\pi, 4\pi, 6\pi$

after division by 4

to convert z^4 to z

If you want can write this out in less clear way e.g.

$$2^{1/8} = 1.0905 \quad e^{i\pi/16} = .981 + i(.195)$$

$$z = 1.068 + .213i \quad \text{is one soln}$$

7-2

6 cont'd

$$z^4 - (3+i)z^2 + (2+i) = 0 \quad \leftarrow \text{quadratic in } z^2$$

Quadratic formula still works

$$\begin{aligned} z^2 &= \frac{[(3+i) \pm \sqrt{(3+i)^2 - 4(2+i)}]}{2} \\ &= \frac{[(3+i) \pm \sqrt{9+6i-1-8-4i}]}{2} \\ &= \frac{[3+i \pm \sqrt{2i}]}{2} \end{aligned}$$

$$\sqrt{2i} = \sqrt{2}e^{i\pi/4} = \sqrt{2}e^{i\pi/4}$$

$$= \sqrt{2}(\cos\pi/4 + i\sin\pi/4) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1+i$$

$$z^2 = \frac{[3+i \pm (1+i)]}{2}$$

$$z^2 = (4+2i)/2 = 2+i$$

- or - $z^2 = 2/z = 1$

$$z^2 = 2+i = \sqrt{5}e^{i\phi} \quad \text{where } \tan\phi = 1/2$$

$$z = 5^{1/4} e^{i\phi/2}; -5^{1/4} e^{i\phi/2} \quad \tan\phi = 1/2$$

$$\theta = 1^\circ - 1$$

The 4
solutions
of
degree 4
eq

8-1

7

$$\ln(3+4i) = \ln 5 \left(\frac{3}{5} + \frac{4}{5}i\right)$$

$$= \ln 5 + \ln e^{i\theta}$$

$$\tan \theta = 4/3$$

$$= \ln 5 + i \tan^{-1}\left(\frac{4}{3}\right)$$

