

Partial diff eqn

Diff eqn

$$\frac{d^2y(t)}{dt^2} = -g_0$$

Q:

What problem is this

function y
one indep
variable (t)

$$y = y_0 + v_0 t - \frac{1}{2} g_0 t^2$$

Q:

what v_0

$$\frac{d^2x}{dt^2} = -kx$$

$$x = A \cos \omega t$$

function x
of one indep
variable t .

$$\omega = \sqrt{\frac{k}{m}}$$

$$+ \frac{v_0}{\omega} \sin \omega t$$

PDE Function f \leq indep variables

$$\psi(x, t) \quad \text{in QM}$$

$$E(x, y, z, t) \quad \text{in EM}$$

Example

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial t}$$

true for all x, t

Q: Sol'n? my $\psi = x + t^2$
something

$$t^2 = 2(2x + t)$$

$$e^{2x} e^t$$

works

PDEI - 2

guess $g = \ln$ $f(x,t) = f(x)g(t)$

$$f'(x)g(t) = 2f(x)g'(t)$$

$$\frac{f''(x)}{f(x)} = \frac{2g'(t)}{g(t)} = a$$

$$f(x) = e^{ax}$$

$$g(t) = e^{\frac{a}{2}t}$$

Family of solns

\uparrow
must be constant
because a or r is
true for all x, t
If you fix t and
vary x $g'(t)/g(t)$

will stay same but

$f''(x)/f(x)$ will vary
cannot be true unless
 $f'''(x)/f(x)$ is in fact
fixed also!

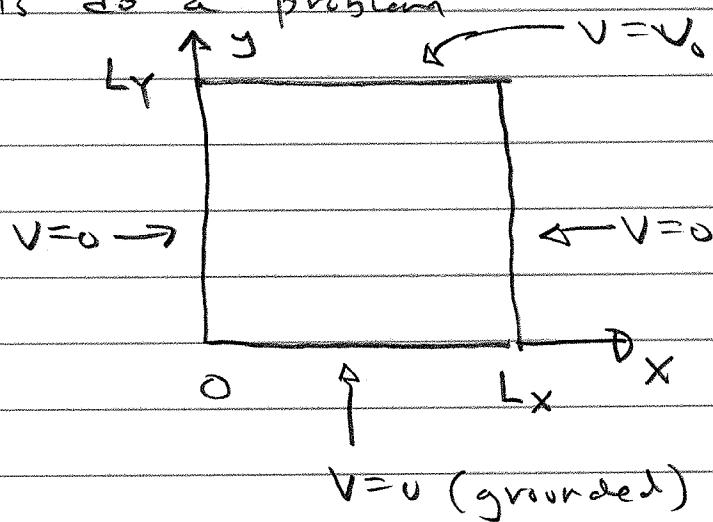
In EM the potential $V(x,y,z)$ obeys

$$\nabla^2 V(x,y,z) = \rho(x,y,z)/\epsilon_0$$

↑
charge density
if $\rho=0$

Recall V determines electric field $\vec{E} = -\vec{\nabla}V$
(in the absence of time dependence)

Let's do a problem



Compute $V(x,y)$ in side box.

As with wave eqn, let's try separation of variables

$$V(x,y) = f(x)g(y)$$

$$f''(x)g(y) + f(x)g''(y) = 0$$

$$\frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)}$$

Contrast this with wave eqn

$$\frac{f''(x)}{f(x)} = \frac{1}{V^2} \frac{g''(t)}{g(t)}$$

Same sign here for wave eqn.

Even so, idea is same that ratio must be a constant

$$\frac{f'(x)}{f(x)} = -\frac{g''(y)}{g(y)} = -k^2$$

$$f(x) = \sin kx \cosh ky$$

$$g(y) = \sinh ky \cosh ky$$

Eliminate $\cosh ky$ from $f(x)$ and also $k = \frac{n\pi}{L_x}$

To obey boundary conditions $f(x=0) = f(x=L_x) = 0$

Similarly eliminate $\cosh ky$ from $g(y)$

because $g(y=0) = 0$.

Thus our solns are

$$\sin \frac{n\pi}{L_x} x \sinh \frac{n\pi}{L_x} y$$

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$$V(x,y) = \sum_n a_n \sin \frac{n\pi x}{L_x} \sinh \frac{n\pi y}{L_x}$$

Q: What to do now?

A: we have not used $V(x, y=L_y) = V_0$

$$\ast \quad V_0 = \sum_n a_n \sin \frac{n\pi x}{L_x} \sinh \frac{n\pi L_y}{L_x}$$

Recall basic idea of FT

$$\int_0^{L_x} \sin \frac{n\pi x}{L_x} \sin \frac{m\pi x}{L_x} dx = 0 \quad \text{if } n \neq m$$



$$= \frac{L_x}{2} \quad \text{if } n = m$$

In words, The functions $\sin \frac{n\pi x}{L_x}$

are orthogonal?

So multiplying \ast by $\sin \frac{m\pi x}{L_x}$ and integrating

$$\int_0^{L_x} V_0 \sin \frac{m\pi x}{L_x} dx = a_m \frac{L_x}{2} \sinh \frac{m\pi L_y}{L_x}$$

(I am just redoing something Anatoliy did for

you last week — deriving formula for a_m)

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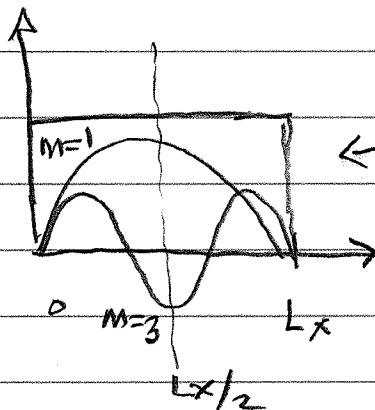
$$\text{LHS} \quad \int_0^{Lx} V_0 \sin \frac{m\pi x}{Lx} dx = -V_0 \frac{Lx}{m\pi} \cos \frac{m\pi x}{Lx} \Big|_0^{Lx}$$

$$= -\frac{V_0 Lx}{m\pi} \left\{ (-1)^m - 1 \right\}$$

$$= \frac{V_0 Lx}{m\pi} \left\{ 1 - (-1)^m \right\}$$

\curvearrowleft 2 if m is odd
 \curvearrowright 0 if m is even

"obvious"
by symmetry



\leftarrow odd m are useful

but even m

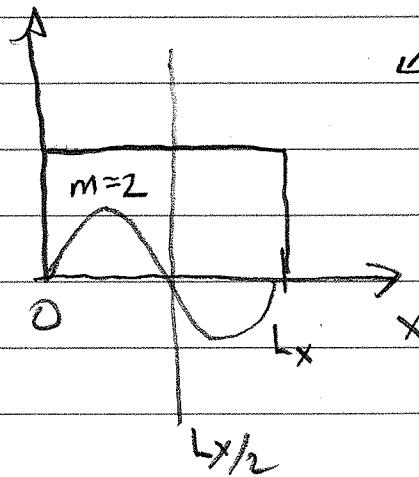
have wrong
symmetry about

$Lx/2$

Thus

$$q_m = \frac{2V_0 Lx}{m\pi} \frac{2}{Lx} \frac{1}{\sinh \frac{m\pi Lx}{Lx}}$$

(for m odd)



$$V(x,y) = \sum_{n \text{ odd}} \frac{4V_0}{n\pi} \frac{1}{\sinh \frac{n\pi Ly}{Lx}} \sin \frac{n\pi x}{Lx} \sinh \frac{n\pi y}{Lx} ?$$

Numerical aside:

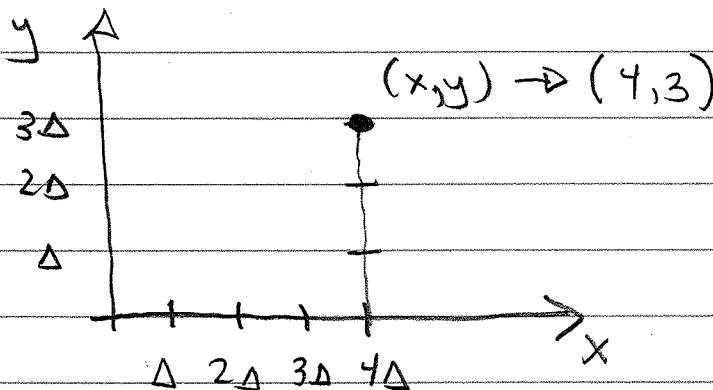
Sol'n of these sorts of EM problem with computer is very easy. (i.e. Poisson eqn or Laplace Eqn)

$$-\frac{\partial^2}{\partial x^2} V(x, y) - \frac{\partial^2}{\partial y^2} V(x, y) = g(x, y)/\epsilon_0$$

Discretize space in units of Δ , That is

$$x = n\Delta \quad V(x, y) \rightarrow V[n][m]$$

$$y = m\Delta$$



First derivative $f'(x) = \frac{f(x+\Delta) - f(x)}{\Delta} = \frac{f[n+1] - f[n]}{\Delta}$

Q: Second derivative ?

$$f(x+\Delta) = f(x) + \Delta f'(x) + \frac{1}{2} \Delta^2 f''(x) + \dots$$

$$f(x-\Delta) = f(x) - \Delta f'(x) + \frac{1}{2} \Delta^2 f''(x) + \dots$$

$$f(x+\Delta) + f(x-\Delta) = 2f(x) + \Delta^2 f''(x)$$

$$f''(x) = \frac{1}{\Delta^2} [f(x+\Delta) - 2f(x) + f(x-\Delta)]$$

$$f''[n] = \frac{1}{\Delta^2} [f[n+1] - 2f[n] + f[n-1]]$$

Thus, Poisson Eqn is

$$\frac{\partial^2}{\partial x^2} \rightarrow v[n+1][m] - 2v[n][m] + v[n-1][m]$$

$$\frac{\partial^2}{\partial y^2} \rightarrow + v[n][m+1] - 2v[n][m] + v[n][m-1]$$

$$= -\Delta^2 \rho[n][m]/\epsilon_0$$

"Solve" for $v[n][m]$

$$v[n][m] = \left\{ v[n+1][m] + v[n-1][m] + v[n][m+1] + v[n][m-1] \right\} / 4$$

$$+ \Delta^2 \rho[n][m]/\epsilon_0$$

average v
over 4

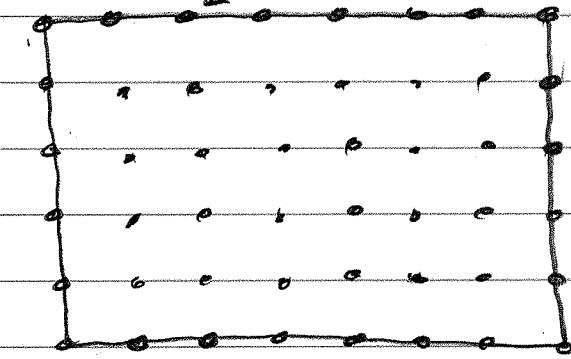
neighboring
points

charge density
contribution

Then just iterate this many times!

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$$v[1][5] = v[2][5] = \dots = v_0$$



$$v[0][0] = v[1][0] = v[2][0] = \dots = 0$$

Can start $v[n][n] = 0$ in interior if you like.

Then iterate many times..