AFS-1

Applications of Fourier Series

(1) To quantum mechanics.

Schroedinger eqn

$$i\hbar \frac{\partial}{\partial t} \psi(x t) = \hat{H} \psi(x t)$$

$$\psi(x t) = e^{-i\hat{H}t/\hbar} \psi(x 0)$$

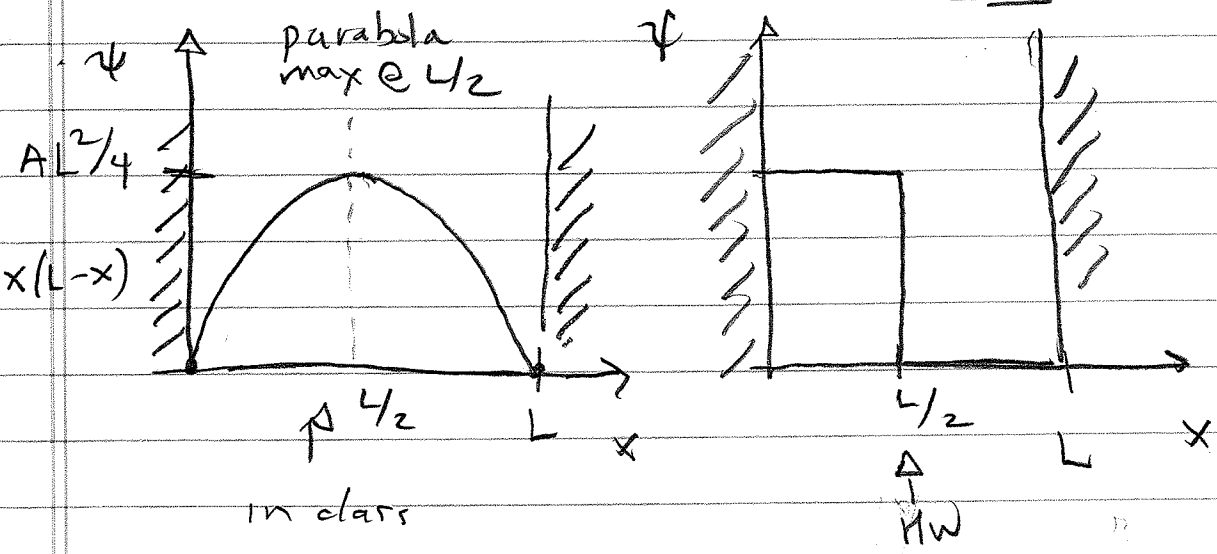
Expand $\psi(x 0) = \sum a_n \phi_n(x)$

eigenstates of $\hat{H}$

$$\psi(x t) = \sum a_n e^{-iE_n t/\hbar} \phi_n(x)$$

operator $\hat{H}$ replaced by <u>number</u> E_n !

Consider ∞ square well:



AFS -2

First, normalize $\psi(x,0) = A\,x(L-x)$

$$1 = \int_0^L |\psi(x,0)|^2\,dx = \int_0^L A^2 x^2 (L-x)^2\,dx$$

$$= A^2 \int_0^L \left(x^2L^2 - 2Lx^3 + x^4\right)dx$$

$$= A^2 \left(x^3L^2/3 - x^4L/4 + x^5/5\right)\Big|_0^L$$

$$= A^2 (L^5)\left(1/3 - 1/4 + 1/5\right) = A^2\, 17L^5/60$$

$$A = \sqrt{60/17L^5}$$

Next, determine $\phi_n(x)$ and E_n:

Inside the infinite square well $V(x) = 0$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

we did this before!

$$\hat{H}\phi_n(x) = E_n\phi_n(x)$$

has solutions $\phi_n(x) = \sqrt{\frac{2}{L}}\sin k_n x$ $\qquad E = \frac{\hbar^2 k^2}{2m}$

normalized

$k_n = \frac{n\pi}{L}$

guarantees $\phi_n(L) = 0$

We now recognize our expansion of $\psi(x,0)$ in $f_n(x)$ as just a Fourier series

$$Ax(L-x) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{L}} \sin k_n x$$

multiply by $\sqrt{\frac{2}{L}} \sin k_m x$ and integrate

$$\sqrt{\frac{2}{L}} \int_0^L Ax(L-x) \sin k_m x \, dx = a_m$$

$$a_m = \sqrt{\frac{2}{L}} \int_0^L Ax(L-x) \sin \frac{m\pi x}{L} \, dx$$

You learned how to do this integral in Math 21 by integration by parts using sine, cosine. It is somewhat easier using complex exponentials

Using results of pages AFS-3A,B

$$\sqrt{\frac{L}{2}} a_m = AL \int_0^L x \sin \frac{m\pi x}{L} dx - A \int_0^L x^2 \sin \frac{m\pi x}{L} dx$$

$$= AL \left\{ -\frac{L}{m\pi} x \cos \frac{m\pi x}{L} + \left(\frac{L}{m\pi}\right)^2 \sin \frac{m\pi x}{L} \right\}_0^L$$

$$- A \left\{ -\frac{Lx^2}{m\pi} \cos \frac{m\pi x}{L} + 2\left(\frac{L}{m\pi}\right)^2 x \sin \frac{m\pi x}{L} + 2\left(\frac{L}{m\pi}\right)^3 \cos \frac{m\pi x}{L} \right\}_0^L$$

AFS-3A

$$\int x \sin kx \, dx$$

$$= \mathrm{Im} \int \underset{v}{x} \underbrace{e^{ikx} dx}_{du}$$

$$= \mathrm{Im} \left\{ \underset{v}{\frac{x e^{ikx}}{ik}} - \int \underset{u}{\frac{e^{ikx}}{ik}} \underset{dv}{dx} \right\}$$

$$= \mathrm{Im} \left\{ -i \frac{x e^{ikx}}{k} - \frac{e^{ikx}}{(ik)^2} \right\}$$

$+ e^{ikx}/k^2$

$e^{ikx} = \cos kx + i \sin kx$

$$= -\frac{x \cos kx}{k} + \frac{\sin kx}{k^2}$$

Check by differentiating

$$\frac{d}{dx}\left(\frac{-x \cos kx}{k} + \frac{\sin kx}{k^2} \right) = -\frac{\cos kx}{k} + x \sin kx + \frac{\cos kx}{k} \checkmark$$

AFS-3B

Similarly $\int x^2 \sin kx \, dx$

$$= \operatorname{Im} \int \underset{\substack{\uparrow \\ v}}{x^2} \underset{\substack{\uparrow \\ du}}{e^{ikx}} \, dx$$

$$= \operatorname{Im} \left\{ x^2 \frac{e^{ikx}}{ik} - \int \frac{e^{ikx}}{ik} 2x \, dx \right\}$$

$$= \operatorname{Im} \left\{ -i \frac{x^2}{k} e^{ikx} + \frac{2i}{k} \int x e^{ikx} dx \right\}$$

$$= -\frac{x^2}{k} \cos kx + \frac{2}{k} \operatorname{Re} \int x e^{ikx} dx$$

$$= -\frac{x^2}{k} \cos kx + \frac{2}{k} \left\{ \frac{x}{k} \sin kx + \frac{\cos kx}{k^2} \right\}$$

from AFS-3A

$$= -\frac{x^2}{k} \cos kx + \frac{2x}{k^2} \sin kx + \frac{2 \cos kx}{k^3}$$

check by differentiation:

$$\overset{\checkmark\checkmark}{-\frac{2x}{k} \cos kx} + \overset{\checkmark}{x^2 \sin kx} + \overset{\checkmark\checkmark}{\frac{2}{k^2} \sin kx} + \overset{\checkmark}{\frac{2x}{k^2} \cos kx} - \frac{2 \sin kx}{k^2}$$

$\boxtimes$

AFS-4

$$\cos(m\pi) = (-1)^m$$

$$\sqrt{\frac{L}{2}}\, a_m = AL\left\{-\frac{L^2}{m\pi}(-1)^m\right\}$$

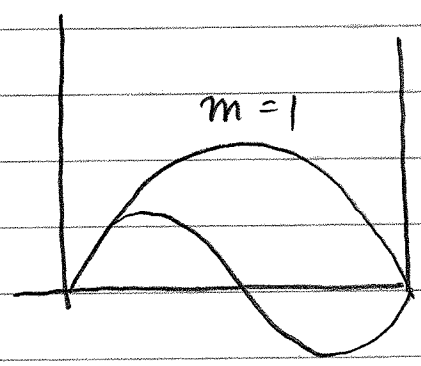
$$-A\left\{-\frac{L^3}{m\pi}(-1)^m + 2\left(\frac{L}{m\pi}\right)^3\left((-1)^m - 1\right)\right\}$$

Terms ① and ② cancel, leaving

$$a_m = \sqrt{\frac{2}{L}}\, 2A\left(\frac{L}{m\pi}\right)^3\left\{1-(-1)^m\right\}$$

This is nonzero only for $m = 1, 3, 5, \ldots$

as expected



← wrong symmetry!

also falls off rapidly with m.

This is because parabola looks almost like $m=1$ term itself!

Final result:

$$\psi(x,t) = \sum_{m=\text{odd}} 4A\left(\frac{L}{m\pi}\right)^3 e^{-i\frac{\hbar^2 m^2 \pi^2}{L^2}t}\, \frac{2}{L}\sin\frac{m\pi x}{L}$$

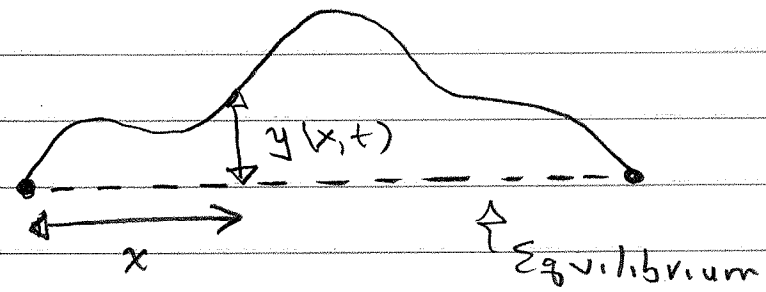
$$A = \sqrt{60/17L^5}$$

Check units: Expect $\psi \sim 1/\sqrt{L}$ and it is $L^3/L^{5/2}\ 1/L$ ✓

AFS-5

(2) To wave equation

If you stretch a string from equilibrium

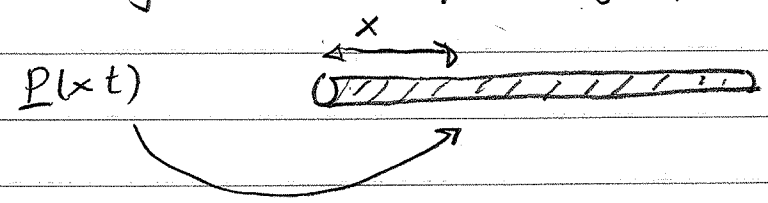


the displacement $y(x,t)$ obeys

$\boxed{Q:}$

$$\partial^2 y/\partial x^2 - 1/v^2 \, \partial^2 y/\partial t^2 = 0$$

(Note: Many other examples eg pressure in a pipe



etc etc)

$\boxed{Q:}$ What is v?

A: $v = \sqrt{T/\mu}$

T = tension

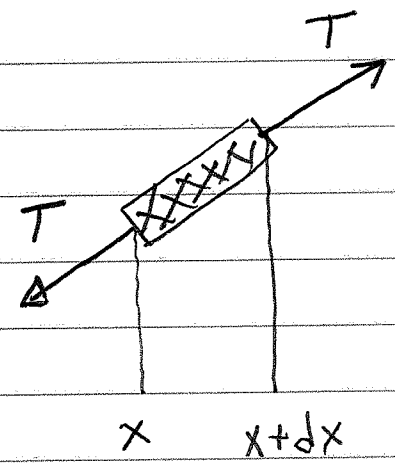
μ = mass/length

$\boxed{Q:}$ What other examples do you know?

A: EM waves!

AFS-5A

$\boxed{Q}$ Does anyone recall derivation of wave eqn for vibrating string?



piece of string mass μdx

F_y at x is

$$T \, dy/dx \Big|_x$$

and at $x+dx$

$$T \, dy/dx \Big|_{x+dx}$$

for any f $\quad f(x+dx) = f(x) + \frac{\partial}{\partial x} f \, dx$

apply to $f = dy/dx$

$$dy/dx \Big|_{x+dx} = dy/dx \Big|_x + \frac{\partial^2 y}{\partial x^2} dx$$

So net force in y direction is

$$\underbrace{T \, dx \frac{\partial^2 y}{\partial x^2}}_{F_{net}} = \underbrace{\mu dx}_{m} \underbrace{\partial^2 y/\partial t^2}_{a}$$

$$\partial^2 y/\partial x^2 = 1/(T/\mu) \, \partial^2 y/\partial t^2$$

AFS-5B

$\boxed{Q}$ Does anyone recall derivation of wave eqn in E+M? This is trickier. First, wave eqn in 3D is

$$\nabla^2 y(x,t) - \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} = 0$$

Laplacian $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Then use Maxwell eqns

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 = 0 \quad \text{in free space}$$

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{j} + \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

and a vector identity

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) + \nabla^2\vec{A}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$-\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) + \nabla^2\vec{E}$$

$$\nabla^2\vec{E} = \mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

AFS-6

separation of variables

$$y(xt) = f(x)g(t)$$

$$f''(x)g(t) = \frac{1}{v^2} f(x) g''(t)$$

$$\underbrace{\frac{f''(x)}{f(x)}}_{\text{function of } x} = \frac{1}{v^2} \underbrace{\frac{g''(t)}{g(t)}}_{\text{function of } t}$$

The only way this can happen is if both are constants

$$\frac{f''(x)}{f(x)} = -k^2 = \frac{1}{v^2} \frac{g''(t)}{g(t)}$$

$f(x) = \sin kx \quad \cos kx$

$g(t) = \sin kvt \quad \cos kvt$

usual notation $\omega = kv$ (eg for light $\omega = ck$)

If we insist $y(0,t) = 0$, $y(L,t) = 0$ (string fixed down at ends)

Just as in our QM problem we have only

$f(x) = \sin m\pi x/L$ as possible $f(x)$.

AFS-7

$\cos kx$ eliminated by $y(0t)=0$

and $k \to m\pi x/L$ only by $y(L,t)=0$

$$y(x\,t) = \sum_{m=1}^{\infty} \sin\frac{m\pi x}{L}\left\{ a_m \cos\frac{m\pi v t}{L} + b_m \sin\frac{m\pi v t}{L}\right\}$$

If the string starts from rest $\partial y/\partial t = 0$ @ $t=0$

then all the $b_m = 0$ because

$$\partial y/\partial t\Big|_{t=0} = \sum_{m=1}^{\infty} \sin\frac{m\pi x}{L}\left\{\frac{m\pi v}{L} b_m\right\}$$

Starting from rest

$$y(x,t) = \sum_{m=1}^{\infty} a_m \sin\frac{m\pi x}{L}\cos\frac{m\pi v t}{L}$$

$$y(x\,0) = \sum_{m=1}^{\infty} a_m \sin\frac{m\pi x}{L}$$

multiply by $\sin\frac{n\pi x}{L}$ and integrate

$$\sqrt{\frac{2}{L}}\int_0^L y(x,0)\sin\frac{n\pi x}{L}\,dx = a_n$$

AFS-8

Starting from rest

So solving wave eqn for vibrating string is practically identical to Sch Eqn for ∞ square well

Expand	$\psi(x,0) = \sum_m a_m \sqrt{\frac{2}{L}} \sin \frac{m\pi x}{L}$	$y(x,0) = \sum a_m \sin \frac{m\pi x}{L}$
determine a_m	$a_n = \sqrt{\frac{2}{L}} \int_0^L \psi(x,0) \sin \frac{n\pi x}{L} dx$	$a_n = \frac{2}{L} \int_0^L y(x,0) \sin \frac{n\pi x}{L}$
final soln	$\psi(x,t) = \sum_{n=1}^{\infty} a_n e^{-iE_n t/\hbar} \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$ $E_n = \hbar^2/2m \; n^2\pi^2/L^2$	$y(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \cos \frac{n\pi v t}{L}$

Can make look even more similar via

$\cos \omega_n t$

$\omega_n = v \frac{n\pi}{L}$

AFS-84

Really Fancy Stuff are GREEN'S FUNCTIONS

$$y(xt) = \sum_m q_m \sin\frac{m\pi x}{L} \cos\frac{m\pi v t}{L}$$

$$q_m = \frac{2}{L}\int_0^L \sin\frac{m\pi x'}{L}\, y(x'0)\, dx'$$

$$y(xt) = \sum_m \int_0^L \frac{2}{L} \sin\frac{m\pi x'}{L}\, y(x',0)\, dx' \sin\frac{m\pi x}{L} \cos\frac{m\pi v t}{L}$$

$$= \int_0^L dx'\, y(x'0) \left\{ \sum_m \frac{2}{L} \sin\frac{m\pi x'}{L} \sin\frac{m\pi x}{L} \cos\frac{m\pi v t}{L} \right\}$$

This is $G(x, x', t)$
the "greens function"
of the wave eqn

$$y(x,t) = \int_0^L dx'\, y(x',0)\, G(x, x', t)$$

initial $(t=0)$ knowledge of y at position x'

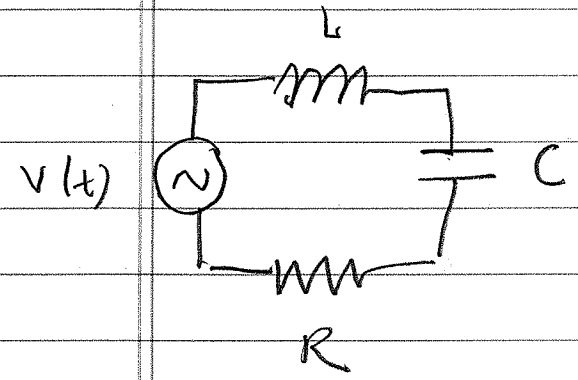
G describes spread of information from x to x' at time t

AFS-9

(3) <u>Example 3</u>: the driven LRC circuit

(or driven mass on spring $\leftrightarrow$ any oscillator)

$$L^2 \frac{dQ}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$



$V(t)$ applied voltage to circuit

Suppose $V(t) = V_0 \cos \omega t$

$$Q(t) = \mathrm{Re}\, \frac{V_0\, e^{i(\omega t - \phi)}}{\sqrt{L^2(\omega_0^2 - \omega^2) + \omega^2 R^2}}$$

$$\tan\phi = \frac{\omega R}{L(\omega_0^2 - \omega^2)}$$

We did this problem in week #1:

$\omega_0^2 = 1/LC$

But what about general driving functions like

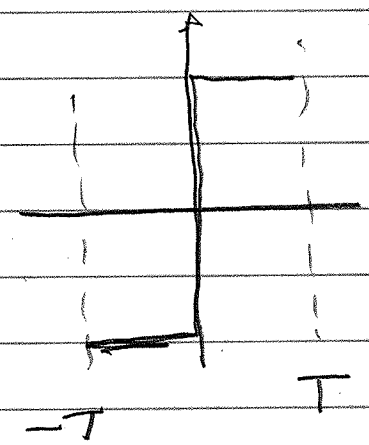
How do we compute the response of the circuit

AFS-10

The key is to take advantage of the fact that the differential eqn is linear. Thus its response to $V_1(t) + V_2(t)$ can be obtained as $Q_1(t) + Q_2(t)$ where Q_1 and Q_2 are responses to V_1 and V_2

So what needs to be done is to write the periodic function $V(t)$ as a Fourier expansion

$\sin \frac{\pi t}{T}$ and $\cos \frac{\pi t}{T}$



This was basically Example #1 of Anatoly's lecture!

Q: Does anyone recall some of those results

A: $\cos \pi t/T$ terms are absent since $V(t)$ is odd

SUMMARY: Almost anywhere you look, solving physics problems involves Fourier series

QM, CM, EM ← EM waves or LRC circuit

particle in an as square well

vibrating string or mass/spring general $F(t)$