Fourier Series

In a finite dimensional vector space it is convinient to choose basis $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ and represent vectors by their components $v_1, \ldots, v_n$ in that basis:

(1) $$\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \ldots + v_n\vec{e}_n = \sum_{k=1}^{n} v_k \vec{e}_k$$

One often chooses an orthogonal basis:

(1') $$(\vec{e}_j, \vec{e}_k) = 0 \quad \text{if } j \neq k$$

Then, multiplying both sides of (1) by $\vec{e}_k$,

(2) $$\underbrace{v_k}_{\text{components of } \vec{v} \text{ in } \{\vec{e}_k\}_{k=1}^{n} \text{ basis}} = \frac{(\vec{v}, \vec{e}_k)}{(\vec{e}_k, \vec{e}_k)}$$

If $(\vec{e}_k, \vec{e}_k) = 1 \ \forall k$, then the basis is orthonormal

The basic idea behind Fourier series is that functions, which live in an infinite dimensional space, can be represented by Fourier coefficients that play a role of "vector components" in this abstract infinite dim. space.

FS-2

Let us define a scalar product of two functions $f(x)$ and $g(x)$ on $[a, b]$ in the usual way:

(3) $$(f, g) = \int_a^b f(x)\, g(x)\, dx$$

We will use this def. even when f and g are not continuous on $[a; b]$, so that the integral is improper

Similarly to (1') we consider an orthogonal system of continuous functions $\{e_k(x)\}_{k=1}^{\infty} \equiv \{\vec{e}_1(x), \vec{e}_2(x), \dots\}$ on $[a, b]$ (orthogonal in a sense of (3), i.e., $\int_a^b e_1(x) e_2(x)\, dx = 0$ and so on).

Then we can at least hope that with certain not too restrictive conditions on $f(x)$, which we'll discuss later, we can write

(4) $$f(x) \underbrace{\sim}_{\text{"represented by"}} \sum_{n=1}^{\infty} \alpha_k e_k(x),$$ (for every $f(x)$ satisfying these conditions)

where, similar to (2) above,

(5) $$\underbrace{\alpha_k}_{\text{Fourier coefficients}} = \frac{(f(x), e_k(x))}{(e_k(x), e_k(x))}; \quad k \in \mathbb{N}$$

These are called Dirichlet conditions

Compare these to (1) and (2)!

There are many sets of orthogonal functions $\{e_k(x)\}_{k=1}^{\infty}$, such as spherical harmonics, hermite polynomials, Bessel functions, etc. Also, later in the course you will learn about Legendre polynomials that are orthogonal in $[-1, 1]$. Here we consider a trigonometric set of $2L$-periodic functions orthogonal in $[-L, L]$:

(6) $$e_0(x) = \frac{1}{2};\ e_1(x) = \sin\frac{\pi x}{L};\ e_2(x) = \cos\frac{\pi x}{L};\ \dots\ e_k(x) = \sin\left(\frac{k\pi x}{L}\right);\ e_{k+1}(x) = \cos\left(\frac{k\pi x}{L}\right), \dots$$

We claim (and leave it to mathematicians to prove) that <u>every</u> suitably well-behaved (i.e. satisfying Dirichlet conditions) $2L$-periodic function can be represented by <u>Fourier series</u>:

(7) $$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_n \cos\left(\frac{k\pi x}{L}\right) + b_n \sin\left(\frac{k\pi x}{L}\right)\right],$$

where a_n, b_n are Fourier coefficients.

We've isolated a_0 for reasons that will become evident later.

!!! Skip this page to avoid mathematical generalities

Dirichlet conditions

1) $f(x)$ is single valued with a finite # of discontinuities in $[L; -L]$.
2) $f(x)$ has finite # of extrema in $[-L, L]$
3) $\int_{-L}^{L} |f(x)|dx$ must converge

Most of functions we are dealing with in physics satisfy these conditions. Mathematicians came up with diabolic examples of functions that do not satisfy DC's, but those are rarely, if ever, used in physical applications.

Before we give a proof of orthogonality of the trigonometric set (6) and how to find Fourier coefficients, let's make some remarks:

i) Notice that $f(x)$ are very special functions: $2L$-periodic <u>and</u> satisfying DC's given above. In particular, $f(x)$ can have a finite # of discontinuities. Thus Fourier series are different from Taylor series you learned earlier, they can describe functions that are not everywhere continuous and/or differentiable.

ii) We will not prove this, but the trigonometric set of functions in $[-L, L]$ is complete* in a sense that <u>every</u> function can be expressed as a <u>linear</u> combination of these functs.

iii) We use a symbol $\sim$ rather than $=$ in (7), because Fourier series (FS) can in general diverge or converge not to $f(x)$. We will discuss convergence of FS later when we give some examples.

* in a space of square integrable functions with scalar product defined by (3).

Properties of periodic functions

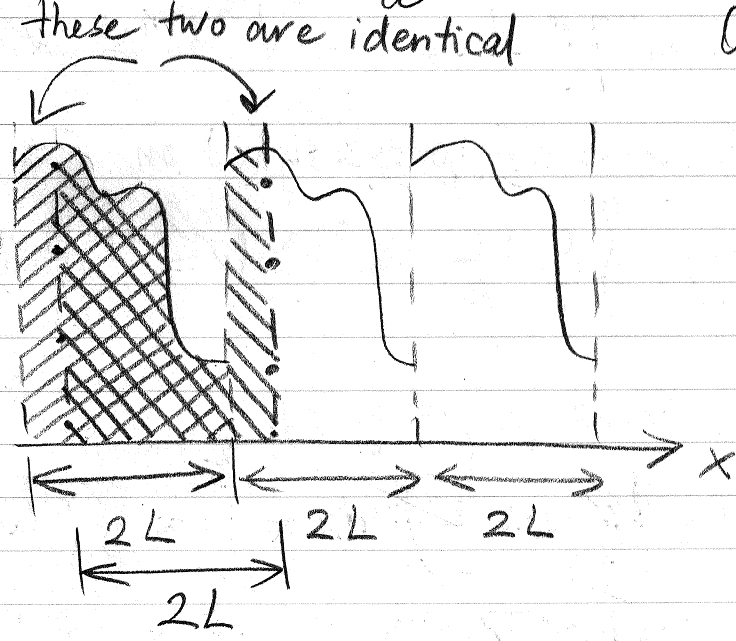
Consider a $2L$-periodic function $f(x)$:

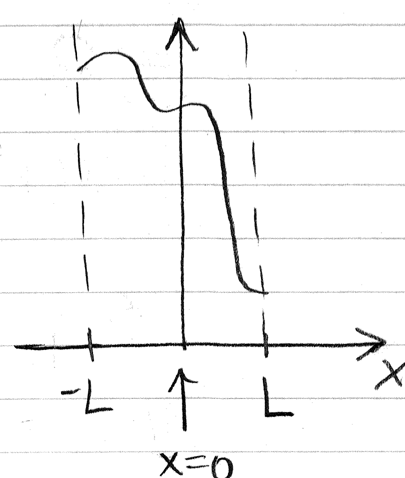
$$f(x+2L) = f(x), \quad \forall x \in \mathbb{R}$$

Property (1) The integral $\int_d^{d+2L} f(x)\,dx$ doesn't depend on d:

One often chooses $d=-L$

A graphical proof





Hence, we can always consider $\int_{-L}^{L} f(x)\,dx$.

Property (2) $\int_{-L}^{L} \sin\left(\frac{k\pi x}{L}\right)dx = \int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right)dx = 0; \ k \in \mathbb{Z}$

Proof: $\int_{-L}^{L} \sin\left(\frac{k\pi x}{L}\right)dx = -\frac{L}{k\pi}\cos\left(\frac{k\pi x}{L}\right)\Big|_{-L}^{L} =$

$= -\frac{L}{k\pi}\left(\cos(k\pi) - \cos(-k\pi)\right) = \{\cos\alpha \text{ is even}\}$

$\underset{L}{=} -\frac{L}{k\pi}\left(\cos(k\pi) - \cos(k\pi)\right) = 0$

$\int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right)dx$ is done analogously.

Now, let's prove that the trigonometric set

$\frac{1}{2}$, $\sin\frac{\pi x}{L}$, $\cos\frac{\pi x}{L}$, ..., $\sin\frac{k\pi x}{L}$, $\cos\frac{k\pi x}{L}$, ...

is orthogonal in $[-L, L]$

Start with $\sin\left(\frac{\pi k x}{L}\right)$ and $\cos\left(\frac{\pi n x}{L}\right)$, $\underline{\underline{n \neq k}}$

$$\int_{-L}^{L} \sin\frac{\pi k x}{L} \cos\frac{\pi n x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left[\sin\frac{\pi(n+k)x}{L} - \sin\frac{\pi(n-k)x}{L}\right]dx$$

$= 0$, because of property (2) on FS-5.

Analogously, $\int_{-L}^{L} \sin\frac{\pi k x}{L} \cos\frac{\pi n x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left[\sin\frac{2\pi n x}{L} - \sin 0\right]dx$

$= 0$. by the same property (2).

Also, $\int_{-L}^{L} \sin\frac{\pi k x}{L} \sin\frac{\pi n x}{L}\,dx = \int \cos\frac{\pi k x}{L} \cos\frac{\pi n x}{L}\,dx = 0$, $(n \neq k)$

but not for $n = k$!

Finally $\int_{-L}^{L} \frac{1}{2} \cdot \sin\frac{\pi n x}{L}\,dx = \int_{-L} \frac{1}{2} \cdot \cos\frac{\pi n x}{L}\,dx = 0$,

by the same property (2).

$n = k$ case!

$\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

$$\int_{-L}^{L} \cos\frac{\pi n x}{L} \cos\frac{\pi n x}{L}\,dx = \underbrace{\frac{1}{2}\int_{-L}^{L} \cos\frac{2\pi n x}{L}\,dx}_{=0 \text{ by property (2)}} + \frac{1}{2}\int_{-L}^{L} dx = \frac{1}{2} 2L = \underline{\underline{L}}$$

Analogously, $\int_{-L}^{L} \sin\frac{\pi n x}{L} \sin\frac{\pi n x}{L}\,dx = L$

Thus, for suitably well-behaved $2L$-periodic functions

(7) $$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L}\right)$$

How do we find a_n and b_n?

We could, of course, use (5) and simplify our work. But let's do it explicitly. The idea is to multiply both sides by $\cos\left(\frac{m\pi x}{L}\right)$, integrate over a period and use orthogonality of the trigonometric set.

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \int_{-L}^{L}\cos\frac{m\pi x}{L}\left\{\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos\frac{k\pi x}{L} + b_k\sin\frac{k\pi x}{L}\right\}dx$$

$$= \begin{cases} m=0; & \frac{a_0}{2}\int_{-L}^{L} dx = La_0 \quad \text{(all other integrals vanish, because of the property (2)} \\ m\neq 0; & a_m\int_{-L}^{L}\cos^2\left(\frac{m\pi x}{L}\right)dx = \quad \text{all other vanish due to the orthogonality} \end{cases}$$

$$= \frac{a_m}{2}\int_{-L}^{L} dx + \frac{a_m}{2}\cancel{\int_{-L}^{L}\cos\left(\frac{2m\pi x}{L}\right)dx}^{\,0} = a_m \cdot L$$

rename $m \to k$ (8) Hence, $$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx; \quad \boxed{a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi x}{L}\,dx}$$

Now you see the reason for $\frac{1}{2}$ in front of a_0 in (7).

With it the formula for a_m works for any k, including 0.

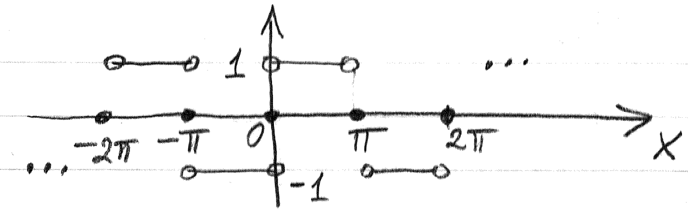
Analogously,

$$\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx = \int_{-L}^{L}\sin\frac{m\pi x}{L}\left\{\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos\frac{k\pi x}{L} + b_k\sin\frac{k\pi x}{L}\right\}dx$$

$$= b_k\int_{-L}^{L}\sin^2\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}b_m\int_{-L}^{L} dx - \frac{1}{2}\cancel{\int_{-L}^{L}\cos\left(\frac{2m\pi x}{L}\right)dx}^{\,0} = b_m \cdot L$$

(9) $$\boxed{b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}\,dx}$$

Example 1



$2L = 2\pi$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0 \qquad f(x) \text{ is an odd function!}$$

$$b_k = \frac{1}{\pi}\int_{-L}^{L} f(x)\sin kx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin kx\,dx + \int_{0}^{\pi} 1\cdot \sin kx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin kx\,dx + \int_0^{\pi}\sin kx\,dx\right] = \frac{2}{\pi}\int_0^{\pi}\sin kx\,dx =$$

$$= -\frac{2}{\pi k}\cos kx\,\Big|_0^{\pi} = -\frac{2}{\pi k}(\cos k\pi - 1) = \frac{2}{\pi k}(1-\cos k\pi) =$$

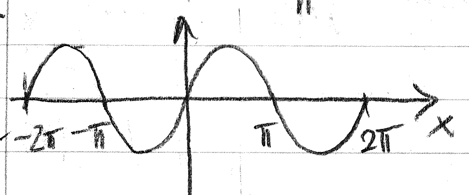
$$= \begin{cases} 0, & k = 2n. \\ \frac{4}{\pi(2n+1)}, & k = 2n+1. \end{cases} \qquad \text{Hence } f(x) \sim \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}$$

$$f(x) \sim \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right)$$

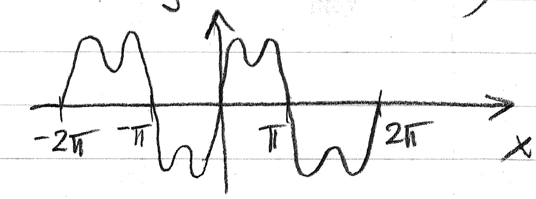
Taking $x = \frac{\pi}{2}$, $f(x) = 1$, one has $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{(-1)^n}{2n+1}$

A lot of these will pop up for free

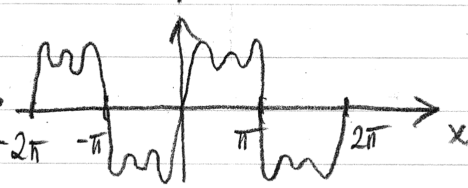
1 term



2 terms



3 terms



We see that at points where $f(x)$ is continuous FS converges to $f(x)$, but at discontinuities there are large oscillations (that will not go away if you take more and more terms in FS of $f(x)$), known as

<u>Gibbs phenomenon</u>.

FS-9

f(x) has to have one-sided derivatives at discontinuities

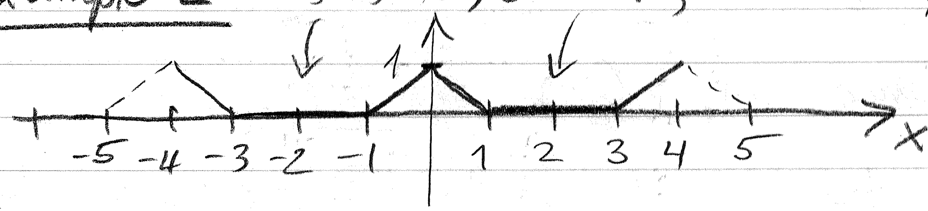
At discontinuities FS converges to

$$\frac{f(x_0+0)+f(x_0-0)}{2},$$

where $f(x_0+0)$ and $f(x_0-0)$ are one-sided limits of $f(x)$ at x_0.

In our example, for $x_0=0$, $f(0+0)=+1$, $f(0-0)=-1$ and FS converges to $\frac{1+(-1)}{2}=0$.

__Example 2__ $f(x)=0, -3\le x\le -1$, $1\le x\le 3$, ...



Period $=2L=4$; $L=2$

do this separately from other a_k's $\rightarrow$ $a_0=\frac{1}{2}\int_{-2}^{2} f(x)\,dx=\frac{2}{2}\int_0^2 f(x)\,dx=\int_0^1 (1-x)\,dx=\frac{1}{2}$

All $b_k=0$ ($f(x)$ is even, so no sines)

For $k\neq 0$: $a_k=\frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{kx\pi}{2}\,dx=$

$=\frac{2}{2}\int_0^2 f(x)\cos\frac{kx\pi}{2}\,dx=\int_0^1 (1-x)\cos\frac{kx\pi}{2}\,dx=$

$=\left\{\int u\,dv=uv-\int v\,du,\ u=1-x;\ dv=\cos\frac{kx\pi}{2}\,dx \quad du=-1\right\}$

$=(1-x)\frac{2}{\pi k}\sin\frac{k\pi x}{2}\Big|_0^1+\frac{2}{\pi k}\int_0^1 \sin\frac{\pi k x}{2}\,dx=$

$=0+\left(\frac{2}{\pi k}\right)^2\left(-\cos\frac{k\pi x}{2}\Big|_0^1\right)=\left(\frac{2}{\pi k}\right)^2\left(1-\cos\frac{k\pi}{2}\right)$

No discontinuities $\Rightarrow$ FS of $f(x)$ converges to $f(x)$ at every $x\in\mathbb{R}$

$$f(x)\sim\frac{1}{4}+\frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\left(1-\cos\frac{k\pi}{2}\right)}{k^2}\cos\frac{k\pi x}{2}.$$

In general, FS for even and odd functions

Even function: $f(x) = f(-x)$

(10) $$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$

Odd function: $f(x) = -f(-x)$

(11) $$\int_{-L}^{L} f(x)\,dx = 0$$

Hence if $f(x)$ is even

(12)

$$b_K = \frac{1}{L}\int_{-L}^{L} \underset{\uparrow\ \text{even}}{f(x)}\, \underset{\uparrow\ \text{odd}}{\sin}\frac{K\pi x}{L}\,dx = 0, \text{ by (11)}$$

$$a_K = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{K\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{K\pi x}{L}\,dx, \text{ by(10)}$$

If $f(x)$ is odd

(13)

$$a_K = \frac{1}{L}\int_{-L}^{L} \underset{\uparrow\ \text{odd}}{f(x)}\, \underset{\uparrow\ \text{even}}{\cos}\frac{K\pi x}{L}\,dx = 0, \text{ by (11)}$$

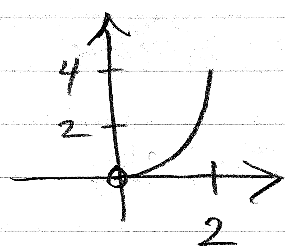
$$b_K = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{K\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{K\pi x}{L}\,dx,$$

by (10)

Non-periodic functions

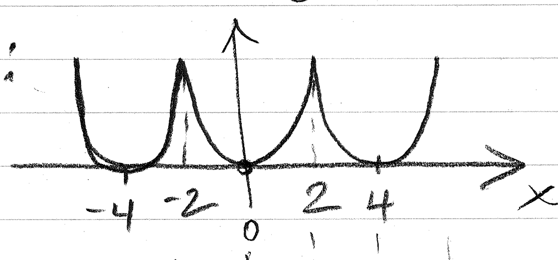
If we wish to find the FS of a non-periodic function (only within a fixed range), we can continue this function outside the range to make it periodic. This can be done in different ways. (see example below). The FS found this way will correctly represent the function in the range (except, maybe, at the endpoints).

Example $f(x) = x^2 \; ; \; 0 < x \leq 2$



Can continue this function to make it even or odd (or neither!)

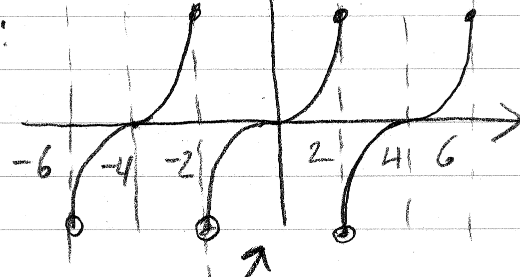
Even:



$f(x+4) = f(x)$

$2L = 4$

Odd:



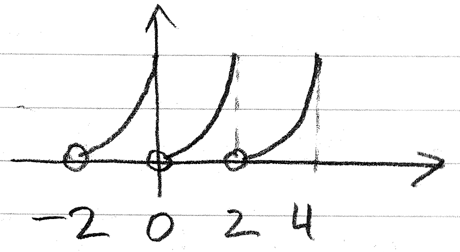
$f(x+4) = f(x)$

$2L = 4$

Has discontinuities!

FS of this will converge to $\frac{4-4}{2} = 0$ at discontinuities.

FS-12



Niether even, nor odd

$$f(x+2) = f(x)$$
$$2L = 2$$

Take the <u>even case</u>

Using (12) $a_0 = \frac{2}{2}\int_0^2 x^2 dx = \frac{x^3}{2}\Big|_0^2 = \frac{8}{3}.$

$k \neq 0$: $a_k = \frac{2}{2}\int_0^2 x^2 \cos\frac{k\pi x}{2} dx = \{$ taking by parts twice ... $\} =$

$$= \frac{16}{\pi^2 k^2}\cos k\pi = \frac{16}{\pi^2 k^2}(-1)^k$$

Hence $x^2 = \frac{4}{3} + \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k \cos\left(\frac{k\pi x}{2}\right)}{k^2}$; $\underbrace{0 < x \leq 2}_{\text{only in this range!}}$

Non-periodic well-behaved functions in general can be represented by <u>Fourier integrals</u>

FS-13

Complex form of FS

Let's simplify things and take $L=\pi$ case

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$\cos kx = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right); \quad \sin kx = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$$

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{a_k}{2}\left(e^{ikx} + e^{-ikx}\right) + \frac{b_k}{2i}\left(e^{+ikx} - e^{-ikx}\right)$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} \underbrace{\frac{a_k - i b_k}{2}}_{c_k} e^{ikx} + \underbrace{\frac{a_k + i b_k}{2}}_{c_{-k}} e^{-ikx} \qquad k = 0, 1, \ldots$$

Notice that $c_k^* = c_{-k}$.

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left(\cos kx - i \sin kx\right) dx =$$

this comes from $\frac{a_k - i b_k}{2}$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

$$c_{-k} = c_k^* = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ikx} dx$$

Thus $$f(x) \sim \sum_{k=-\infty}^{+\infty} c_k e^{ikx}; \quad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

In general, for $2L$-periodic function

$$f(x) \sim \sum_{k=-\infty}^{+\infty} c_k e^{ikx}; \quad c_k = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-ikx} dx.$$