## The Gambler's Ruin Problem

Consider the problem of two gamblers. The first has a bankroll of $M_1$ dollars and the second has $M_2$ dollars. They play a fair game (probability $p = 1/2$ of each one winning), and bet one dollar on each game. We will show here that the probability that the first gambler eventually goes bankrupt is $M_2/(M_1+M_2)$ and that the second one eventually goes bankrupt is $M_1/(M_1+M_2)$. A consequence of this is that if the second gambler is "the bank" with very large resources $M_2 \to \infty$ then gambler one is certain to go home broke ($M_2/(M_1+M_2) \to 1$). Another interesting result is that the expected time until one or the other goes broke is $\tau = M_1 M_2/2$.

We will frame this problem as a random walk which starts at the origin $i = 0$. The walk ends when either $i = -M_2$ or $i = +M_1$ is reached. Define $f(i)$ to be the probability that, if the walk is at location $i$, $M_1$ will be reached before $-M_2$. That is, $f(i)$ is the probability that gambler one is ruined before gambler two. We know immediately the value of $f(i)$ for two special locations $i$. We have $f(-M_2) = 0$ and $f(M_1) = 1$. The reason is simple: If we are already at $i = M_1$ the probability that $M_1$ is reached first is equal to one! We are already there! Similarly for $i = -M_2$ we will never reach $M_1$ since we stop immediately at $-M_2$.

We can write a simple equation which $f(i)$ must obey for the rest of the values $i$: If we are at position $i$, there is a probability 1/2 the walk will go to $i+1$ and a probability 1/2 the walk will go to $i-1$. When the walk reaches these new points, the probabilities gambler two is ruined first are $f(i+1)$ and $f(i-1)$. Since these are the only two things that can happen from position $i$, we conclude that

$$f(i) = \frac{1}{2}\, f(i-1) + \frac{1}{2}\, f(i+1) \tag{1}$$

There is a really important hidden assumption here: In writing this equation, we are assuming, in technical mathematical jargon, that the walk is a "Markov" chain. What this means is that when we get to the new locations $i \pm 1$ the probabilities of ruin do not depend on how we got there. That is, we can use the exact same $f$ values at all locations regardless of how we arrive.

It is easy to solve Eq. (1). First show that $f(i) = A\,i + B$ is a solution for any $A, B$.

Then get $A, B$ from the two known values,

$$f(-M_2) = -AM_2 + B = 0 \qquad\qquad f(M_1) = AM_1 + B = 1 \tag{2}$$

Solving yields,

$$A = \frac{1}{M_1 + M_2} \qquad B = \frac{M_2}{M_1 + M_2} \qquad f(i) = \frac{i + M_2}{M_1 + M_2} \tag{3}$$

We can now just plug in $i = 0$ to get the probability that gambler one is ruined first, starting at the origin:

$$f(0) = \frac{M_2}{M_1 + M_2} \tag{4}$$

This is a very pretty (and intuitive) result: The probability gambler one gets ruined first depends on the ratio of her bankroll to that of gambler two. Suppose gambler one has twice as much money as gambler two to start out: $M_1 = 2M_2$. The probability she will go bankrupt is $f(0) = 1/3$. This is less than $1/2$, as expected, since she had more money to start.

We can also ask about the expected time $\tau(i)$ that the game will last if the walk is currently at $i$. Convince yourself that $\tau(i)$ must satisfy,

$$\tau(i) = \frac{1}{2}\,\tau(i-1) + \frac{1}{2}\,\tau(i+1) + 1 \tag{5}$$

Notice this differs from Equation (1) because of the "+1" on the right hand side. Why is this term there? We also know the value of $\tau(i)$ at the two endpoints. Construct an argument that

$$\tau(-M_2) = 0 \qquad\qquad \tau(M_1) = 0 \tag{6}$$

Equations 5-6 can be solved by the guess $\tau(i) = -i^2/2 + C\,i + D$. Show this is true and then prove that

$$\tau(i) = \frac{1}{2}\,(\,M_1 - i\,)\,(\,i + M_2\,) \tag{7}$$

In particular, $\tau(0) = M_1 M_2/2$. This means that even though you have no chance of surviving against the bank if it has infinite resources, the expected time for the game to end is infinite. This is why many students were finding they still had money left even after very long sets of millions of games.

The generalization of these results to an unequal game where there is a probability $p$ of gambler one winning and $q$ of losing each bet (with $p+q=1$) is obtained by using the same basic equation (1), but instead of assuming a solution of the form $f(i) = A\,i + B$, trying a solution of the form $f(i) = A\,r^i + B$. You might enjoy doing this problem as an exercise.