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We want to prove that

$$N = \frac{1}{2^N}\sum_{k=0}^{N}\binom{N}{k}(2k-N)^2.$$

$$\rightarrow \quad N = \frac{1}{2^N}\sum_{k=0}^{N}\binom{N}{k}(4k^2-4kN+N^2)$$

We split this into 3 separate sums so that the RHS is equal to the sum of $S_1+S_2+S_3$.

$$S_1 = \frac{1}{2^N}\sum_{k=0}^{N}\binom{N}{k}N^2 = \frac{N^2}{2^N}\sum_{k=0}^{N}$$

$$S_2 = \frac{1}{2^N}\sum_{k=0}^{N}\binom{N}{k}(-4kN) = -\frac{4N}{2^N}\sum_{k=0}^{N}\binom{N}{k}k$$

$$S_3 = \frac{1}{2^N}\sum_{k=0}^{N}\binom{N}{k}4k^2 = \frac{4}{2^N}\sum_{k=0}^{N}\binom{N}{k}k^2$$

First, we deal with S_1.

$$S_1 = \frac{N^2}{2^N}\sum_{k=0}^{N}\binom{N}{k}.$$

But $\sum_{k=0}^{N}\binom{N}{k} = \binom{N}{0}+\binom{N}{1}+\dots+\binom{N}{N} = 2^N$, so

$$S_1 = \frac{N^2}{2^N}\cdot 2^N = \boxed{N^2.}$$

Next, we deal with S_2. To do this, we need a formula for $\sum_{k=0}^{N}\binom{N}{k}k$. With some work ☆, we find that this is equal to $N\cdot 2^{N-1}$. So S_2 becomes

$$-\frac{4N}{2^N}\cdot N\cdot 2^{N-1} = \boxed{-2N^2}.$$

Lastly, we want to simplify S_3. In order to do this, we need to find a formula for $\sum_{k=0}^{N}\binom{N}{k}k^2$. We guess that this is equal to $N(N+1)2^{N-2}$ from the fact that $N = S_1+S_2+S_3 = N^2-2N^2+S_3 \rightarrow S_3 = N^2+N = \frac{4}{2^N}\underbrace{(N)(N+1)2^{N-2}}_{\sum_{k=0}^{N}\binom{N}{k}k^2}$

Continue on back page.

☆ I can write up the formal proof (let me know), but the main idea behind the proof of $\sum_{k=0}^{N}\binom{N}{k}k = N\cdot 2^{N-1}$ is to pair up the elements since $\binom{N}{k}=\binom{N}{N-k}$ so $k\binom{N}{k}+(N-k)\binom{N}{N-k} = N\binom{N}{k}$, which can be simplified.

We now want to prove that

$$T = \sum_{k=0}^{N} \binom{N}{k} k^2 = N(N+1)\, 2^{N-2}.$$

we start by simplifying the LHS.

$$T = \sum_{k=0}^{N} \binom{N}{k} k^2 = \sum_{k=0}^{N} \frac{N!}{k!\,(N-k)!} \cdot k \cdot k = \sum_{k=0}^{N} \frac{N!}{(k-1)!\,(N-k+1)!} \cdot k(N-k+1) = \sum_{k=0}^{N} \binom{N}{k-1} k (N-k+1),$$

this becomes

$$T = \sum_{k=0}^{N} \binom{N}{k-1} \left((N+1)k - k^2 \right).$$

we can split this up into 2 separate sums (where $T_1 + T_2 = T$).

$$T_1 = \sum_{k=0}^{N} \binom{N}{k-1} (N+1) k = (N+1) \sum_{k=0}^{N} \binom{N}{k-1} k$$

$$T_2 = \sum_{k=0}^{N} \binom{N}{k-1} (-k^2) = - \sum_{k=0}^{N} \binom{N}{k-1} k^2$$

Thus, by $T = T_1 + T_2$,

$$T = \sum_{k=0}^{N} \binom{N}{k} k^2 = (N+1) \sum_{k=0}^{N} \binom{N}{k-1} k - \sum_{k=0}^{N} \binom{N}{k-1} k^2$$

Adding $\sum_{k=0}^{N} \binom{N}{k-1} k^2$ to both sides, we get

$$\sum_{k=0}^{N} \left(\binom{N}{k} + \binom{N}{k-1} \right) \cdot k^2 = (N+1) \sum_{k=0}^{N} \binom{N}{k-1} k.$$

But $\binom{N}{k} + \binom{N}{k-1} = \binom{N+1}{k}$! So the LHS simplifies to

$$\sum_{k=0}^{N+1} \binom{N+1}{k} k^2 = (N+1) \sum_{k=0}^{N} \binom{N}{k-1} k + \boxed{(N+1)^2}^{\star\star} \quad \boxed{\text{equation } A}$$

★★ The $(N+1)^2$ is necessary on the RHS bc we added $(N+1)^2$ also to the LHS to get $\sum_{k=0}^{N+1} \binom{N+1}{k} k^2$. This is from the fact that $\sum_{k=0}^{N} \binom{N+1}{k} k^2 + \binom{N+1}{N+1}(N+1)^2 = \sum_{k=0}^{N+1} \binom{N+1}{k} k^2$. (Sorry I skipped a step & had to fit it here)

We realize that the LHS is just the $N+1$th analog of what we are trying to prove, so if we can show that the LHS $= (N+1)(N+2)\, 2^{N-1}$, we are done.

Now, with a motivation & goal, we turn to simplifying the RHS, we need to simplify $(N+1) \sum_{k=0}^{N} \binom{N}{k-1} k + (N+1)^2$, so first we simplify $\sum_{k=0}^{N} \binom{N}{k-1} k$.

Expanding, we get

$$\sum_{k=0}^{N} \binom{N}{k-1} k = \binom{N}{0} \cdot 1 + \binom{N}{1} \cdot 2 + \dots + \binom{N}{N-2}(N-1) + \binom{N}{N-1} N,$$

From previous simplification, we know that

$$\sum_{k=0}^{N} \binom{N}{k} k = N \cdot 2^{N-1} = \binom{N}{0} \cdot 0 + \binom{N}{1} \cdot 1 + \dots + \binom{N}{N} N$$

We can subtrat these two equations:

$$X = \sum_{k=0}^{N} \binom{N}{k-1} k = \binom{N}{0}\cdot 1 + \binom{N}{1}\cdot 2 + \binom{N}{2}\cdot 3 + \dots + \binom{N}{N-2}(N-1) + \binom{N}{N-1} N$$

$$- \quad N\cdot 2^{N-1} = \qquad \binom{N}{1}\cdot 1 + \binom{N}{2}\cdot 2 + \dots + \binom{N}{N-2}(N-2) + \binom{N}{N-1}(N-1) + \binom{N}{N} N$$

$$X - N\cdot 2^{N-1} = \binom{N}{0}\cdot 1 + \binom{N}{1} + \binom{N}{2} + \dots + \binom{N}{N-1} - N$$

We manipulate the RHS to get:

$$X - N\cdot 2^{N-1} = \binom{N}{0} + \binom{N}{1} + \dots + \binom{N}{N-1} + \binom{N}{N} - 1 - N.$$

And since $\binom{N}{0} + \binom{N}{1} + \dots + \binom{N}{N} = 2^N$, we get:

$$X - N\cdot 2^{N-1} = 2^N - 1 - N$$

So $\quad X = N\cdot 2^{N-1} + 2^N - N - 1!$

Now, we can go back to the original LHS expression $\left(\begin{matrix}\text{LHS of}\\ \text{eq. A}\end{matrix}\right)$ which was $(N+1)\sum_{k=0}^{N}\binom{N}{k-1}k + (N+1)^2$, and it becomes

$$(N+1)(X) + (N+1)^2 = (N+1)(N\cdot 2^{N-1} + 2^N \cancel{- N - 1} + \cancel{N+1})$$

$$= (N+1)(N\cdot 2^{N-1} + 2^N) = 2^{N-1}(N+1)(N+2) \checkmark$$

We have successfully proven that $\sum_{k=0}^{N+1}\binom{N+1}{k}k^2 = (N+1)(N+2)2^{N-1}$, which is equivalent to $\sum_{k=0}^{N}\binom{N}{k}k^2 = N(N+1)2^{N-2}$.

Now we can go back to S_3:

$$S_3 = \frac{4}{2^N}\sum_{k=0}^{N}\binom{N}{k}k^2 = \frac{4}{2^N}\cdot N(N+1)\cdot \cancel{2^{N-2}} = \boxed{N^2 + N}$$

Adding up $S_1 + S_2 + S_3$, we have

$$\underbrace{N^2}_{S_1} \underbrace{- 2N^2}_{S_2} + \underbrace{N^2 + N}_{S_3} = N.$$

This is just the LHS! so we are done.